Seeing the Mathematical Knowledge of Teachers: A Mathematician's Perspective

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Abstract
We consider the question of how to observe the underlying mathematics in classroom discourse, and measure its depth, by examining a particular episode in a class for middle school teachers. We consider ways in which such an enterprise might throw light on the mathematical knowledge of teachers and lead to productive collaboration between mathematicians and educators.

INTRODUCTION

> Over the past several decades, two refrains have echoed throughout the discourse on teachers' knowledge of mathematics: (1) the mathematical knowledge of U.S. teachers is weak, and (2) the mathematical knowledge needed to enable effective teaching is different from that needed by mathematicians. But efforts to improve our understanding of the mathematical knowledge needed for teaching have lacked an adequate theoretical and empirical basis to guide the connection of mathematical knowledge with the work that teachers need to do (RAND 2003, p. 16).

Discussions of mathematical knowledge for teaching often start from the premise that this knowledge is a sort of applied mathematics, that is, mathematics that is specifically needed for the tasks of teaching. Efforts to develop the 'theoretical and empirical basis' often start with an examination of the work teachers do and proceed to extract the mathematics entailed in this work. In this paper I would like to conduct a thought experiment in pure mathematics, an analysis that starts with a classroom piece of mathematics, rather than a teacher's task. I would like to raise the following questions for discussion: Is there a pure mathematics of the classroom? If so, is the study of it useful in conceptualizing the mathematical knowledge of teachers?

A CLASSROOM EPISODE

In an algebra class for middle school teachers, I presented the following problem:

> The expression
>
> $$0.6\left(\frac{t_1 + t_2 + t_3}{3}\right)$$
>
> is the contribution to a student's final score from three test scores (each out of 100). What is a different way of writing this? Which way should a student use in order to
>
> - calculate the total test contribution to their final grade
> - calculate the effect of getting 10 more points on test 2?

Students had a variety of responses to the first question, including the expression

$$0.2t_1 + 0.2t_2 + 0.2t_3.$$

My intention was for students to see that both this form and the original form were useful in different circumstances, and thus lead them to understand that the notion of simplification is relative to the purpose at hand. However, the conversation took quite a different turn:

Teacher A: I wrote $0.2t_1 + 0.2t_2 + 0.2t_3$ because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.

Teacher B: What do you mean you are not sure? The two expressions are obviously the same!

Teacher A: How you can see that just by looking at them?

Teacher B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distribute the 0.2.

Instructor: How do you know you can move the 3 over? What rule says you can do that?

Teacher B: Isn't it because you only have division and multiplication, so it's the commutative law?

Instructor: But division isn't commutative.

Teacher C: But you can write division as multiplication. Just write it as multiplication by 1/3.

MATHEMATICAL ANALYSIS

Let us leave the discussion now and continue with a mathematical analysis of the problem. The purpose of this analysis is not to describe what the teachers should have known, but to describe the mathematics underneath the surface of the discussion.

As Teacher C mentioned, division by 3 is the same multiplication by 1/3. What do we mean by this? Is this a definition or a theorem? It could be that we have defined the operation of division as the operation of multiplication by the multiplicative inverse, and that 1/3 is by definition the multiplicative inverse of 3, in which case the teacher is simply reiterating the definition of "division by 3". On the other hand, we might interpret the teacher's statement as a verbal description of the equality $a/b = (1/b)a$. We could, in addition, have defined a/b to be the solution to the equation $bx = a$. In this case, the teacher is stating a theorem.

Theorem 1. *Given numbers a and b, with $b \neq 0$, the product of the solution to $bx = 1$ and a is the solution to $bx = a$.*

Proof. Let c be the solution to $bx = 1$. Then, by definition, $bc = 1$. Thus, by the associative law and the properties of 1, $b(ca) = (bc)a = 1a = a$. Therefore, ca is the solution to $bx = a$.

If we use the notation a/b for the solution to $bx = a$, then the theorem tells us that $a/b = (1/b)a$, which could be interpreted as "division by b is the same as multiplication by $1/b$", as Teacher C said.

Once we have established that $(1/b)a = b/a$, then we can reason as follows, using the basic properties of number operations:

$$0.6\left(\frac{t_1 + t_2 + t_3}{3}\right) = 0.6\left(\frac{1}{3}(t_1 + t_2 + t_3)\right) \quad \text{(by Theorem 1)}$$

$$= \left(0.6 \bullet \frac{1}{3}\right)(t_1 + t_2 + t_3) \quad \text{(by the associative law)}$$

$$= \left(\frac{1}{3} \bullet 0.6\right)(t_1 + t_2 + t_3) \quad \text{(by the commutative law)}$$

$$= \left(\frac{0.6}{3}\right)(t_1 + t_2 + t_3) \quad \text{(by Theorem 1)}$$

$$= 0.2(t_1 + t_2 + t_3) \quad (\text{since } 3 \bullet 0.2 = 0.6)$$

$$= 0.2t_1 + 0.2t_2 + 0.2t_3 \quad \text{(by the distributive law)}.$$

WHAT DO WE LEARN FROM THE ANALYSIS?

In the classroom exchange I was intrigued by Teacher A's good intuitive reasoning, combined with an apparent inability to perform the necessary manipulations, contrasted with Teacher B's facility at manipulation yet apparent inability to articulate the justification for it. Taking an "obvious" mathematical manipulation of school algebra and reducing it to the underlying laws of operations struck me as a profound mathematical experience for these teachers, and also validated Teacher A's feeling that it was not at all obvious.

The first thing we learn from the mathematical analysis is that there is good reason for both teachers' reactions to the material. There is a great deal of mathematics lurking beneath the seemingly simple operation of moving the 3. The reduction of this operation to the basic laws of arithmetic leads to a lengthy sequence of deductions. Furthermore, before even starting the reduction we must explore the definitions of our terms, and make decisions about them.

More importantly, the mathematical analysis confirms an impression I had when I was teaching this segment, that a profound mathematical discussion has occurred. Teaching is full of such moments, when you feel that a mathematical iceberg has floated into the room, whose tip is in the curriculum, but whose profundity beneath the water has been apprehended by the students. Two questions present themselves:

1) Is the underlying mathematics of a classroom discussion a meaningful notion?
2) If so, how is it related to mathematical knowledge for teaching?

To be clear about Question 1, I am not asking about the quality of the discourse itself, or the nature of the student thinking, but rather about the purely mathematical ideas inherent in the discussion. Personal experience and discussions with other mathematicians lead me to conjecture that the answer to Question 1 is yes. Let me also propose a more modest version of Question 1, since I have no doubt that one or more versions of it have already been explored by mathematics education researchers

1bis) Is there a distinct notion within the realm of mathematics as practiced by research mathematicians of the underlying mathematics of a classroom discussion?

In a recent lecture at the University of Arizona, Barry Mazur quoted the mathematician I.R. Shafarevich:

> Viewed superficially, mathematics is the result of centuries of effort by many thousands of largely unconnected individuals scattered across continents, centuries and millennia. However the internal logic of its development much more resembles the work of a single intellect developing its thought in a continuous and systematic way, and only using as a means a multiplicity of human individualities—much as in an orchestra playing a symphony written by some composer the theme moves from one instrument to another, so that as soon as one performer is forced to cut short his part, it is taken up by another player, who continues it with due attention to the score (Shafarevich 1982, p. 5).

My Question 1, phrased more idiosyncratically, is: What would a trained mathematical performer hear in classroom discourse?

The episode and analysis I have described above can be considered as a first exercise in answering Question 1. There are pedagogical aspects of the classroom episode that one might consider (the interaction between the students, the role of the teacher in guiding the conversation, the choice of question in the first place). However, in analyzing the episode I attempted to limit myself to purely mathematical aspects, and discover what mathematics was implicit in the discussion, without considering how much of that mathematics was present in the minds of the students or the teacher. It would take many repetitions of this process, many mathematical observations of classroom episodes, to decide if such an analysis is possible in general in a replicable way, and if there is some agreed upon notion of the underlying mathematics in classroom discourse, and an agreed upon assessment of its depth. One would have to be careful in developing this notion, since it is in the very nature of the subject that every piece of mathematics has an iceberg lurking beneath it. The immediacy of the underlying mathematics is an important consideration.

My second question, what is the relation between the underlying mathematics and the mathematical knowledge of teachers, is a matter for collaborative research between mathematicians and educators. I will conclude this paper by offering a couple of questions that seem to me worth exploring.

First, it seems likely that the existence of a substantial mathematical substructure should correspond to moments of difficulty or challenge within the curriculum. It would be interesting to compare observations of the underlying mathematics in classroom discourse with observations of student learning and thinking.

Second, it would be interesting to correlate mathematical observations of classroom discourse with the notions of mathematical habits of mind developed by Cuoco and the notions of way of thinking and way of understanding developed by Harel. Cuoco starts from the practice of mathematics by research mathematicians, and Harel from the observation of mathematics as a product of cognition. How is each of these notions related to the pure mathematics underlying classroom discourse?

CONCLUDING REMARKS

The question of the mathematical content, and mathematical depth, of classroom discourse is often the nub of controversy between mathematicians and mathematics educators about the preparation of teachers, in part because of significant differences in understanding of what the mathematics is and what constitutes mathematical depth. A discussion of how to measure it that engages the expertise of both groups could lead to the adaptation of existing instruments or the production of new instruments. Such a discussion could, as a by-product, lead to professional community-building. The two groups, mathematicians and mathematics educators, often sit facing away from each other, one towards the mathematics and the other towards the students. I hope to stimulate a discussion around questions of observing the underlying mathematics and measuring mathematical depth that will turn them around to face each other.

References

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