Interactions between Philosophy and Didactic of Mathematics.

The case of logic, language and reasoning

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Abstract : It is commonly considered that philosophy of mathematics may provide relevant epistemological references for didactics of mathematics. It is certainly less common to think that didactics of mathematics may address relevant questions to philosophy of mathematics. In this paper, we should like to support the thesis that, concerning the relationship between logic, language and mathematical reasoning in mathematics both aspects are involved.
Résumé : Il est habituel de considérer que la philosophie des mathématiques fournit des références épistémologiques pertinentes pour la didactique des mathématiques. Il est sans doute moins habituel de penser que la didactique des mathématiques peut adresser des questions pertinentes à la philosophie des mathématiques. Dans ce texte, nous voudrions défendre la thèse selon laquelle, en ce qui concerne les questions de relations entre logique, langage et raisonnement mathématique, les deux aspects interagissent.

I. First order logic as an epistemological reference for didactic analysis

It is commonly said that most of people, even scientific students, are illogical. Most often, those who support this claim refer to studies in Psychology showing that human reasoning can't be modeled by the syntactic rules derived of propositional logic (Engel, 1991). In these studies, the focus is mainly on implication, due to the specific role of this connector in deductive reasoning. Studying the difficulties of students to deal with implication, the first author of this paper conjectured that propositional logic were a too poor system to model students reasoning in mathematical classroom. Indeed, in mathematics, it is not the case that one deal only with statements either true or false. According to Vergnaud (1991) the explicit categories for knowledge are not only propositions, but also predicates, that model properties and relationship, and arguments. In order to better understand students' difficulties with implication, to get rid of the illusion of the transparency of this notion (Artigue, 1991), we have led an "epistemological inquiry", leant on history of logic (Blanché, 1970), and concerning the elaboration of the concept of implication on the one hand, of truth on the other hand, and their relationship. In this first part, we will relate briefly 1. What emerge from this inquiry for *implication* (I.1); 2. How the distinction between *truth and validity* stated by

Aristotle appear anew at the beginning of last century and 3. Some consequences for mathematics education.

I.1 What is implication really?

Implication appears already in the ancient Greece under three fundamentally different forms. According with Sextus Empiricus "Philon told that conditional is true when it doesn't begin by true to finish by false; so that there are three manners for this conditional to be true, and one to be false."[1]. We can recognize here the propositional connective: "if p, then q" as it will be defined for example by Wittgenstein (1921) through its truth table. A little bit later, the stoics proposed the two rules of inferences related with conditional that they expressed so: "*If the first, the second; the first; therefore the second*", and "*If the first, the second ; not the second ; therefore not the first*[2]. Aristotle, as for him, introduces the notion of concluding syllogisms, that are true whatever their interpretation, as for example "*if A is true of all B and B is true of all C, then A is true of all C".* In the recent period of renew of logic, Frege (1971), Russel (1903), Wittgenstein (1921), Tarski (1936) and Quine (1950) precise the contour of *implication*. Russell (1903) introduced the generalized conditional "$\forall x\ (P(x) \Rightarrow Q(x))$", where P and Q are predicates[3]. As it is the logical form of most (but not all of course) mathematical theorem, it seems clear that a theory of quantification is necessary for mathematics. Russell explicit the relationship between generalized condition and the propositional connective, that needs to consider each conditional statements "$P(a) \Rightarrow Q(a)$" resulting of the substitution of an element a describing a certain non empty class of objects in the open formula "$P(x) \Rightarrow Q(x)$", where x is a free variable (a place-holder). Wittgenstein (1921), introduced the notion of tautology, that means formula that are true for every possible truth values and that are necessarily interpreted by true statements, and explicated the relationship between tautologies and inference rules. For example, "$(p \wedge (p \Rightarrow q)) \Rightarrow q$) is a tautology associated with the inference rule named *Modus Ponens*. Tarski (1933), introducing a semantic definition of truth, considers the central notion of satisfaction of an open sentence by an element, that allows him to elaborate a recursive definition for the truth of the interpretation of a given formal sentence; it defines then the notion of *model of a formula* as a domain in which the interpretation of the formula is true. Finally, it defines the notion of *logical consequence*: a formula **G** is a logical consequence of a formula F means that every model of F is a model of G; or, as well, the conditional "$F \Rightarrow G$" is true for every interpretation in a non empty domain (Tarski, 1936). As an immediate

[1] From Blanché (1970, p.99). We translate from the French.
[2] These two rules are respectively known as Modus Ponens and Modus Tollens,
[3] That means that they will be interpreted by properties or relationship.

consequence, to prove that a formula *G* is not a logical consequence of a formula *F*, it is enough to find a model of *F* that is not a model of *G*. Quine (1950) offers a magisterial synthesis of all these notions.

I.2 Truth *versus* validity from Aristotle to Quine

As we have seen**,** *implication* is a rather polysemic and complex notion, that can be considered at different level: a conditional sentence in ordinary or scientific contexts, that might be true or false; a theorem in a mathematical theory: a true generalized conditional; a logical theorem in propositional or predicate calculus ; an inference rule. Of course, these different modes do not play the same role in mathematics, and this beyond the question of the place of predicate calculus in, or out of, mathematics. Arstotle, in First Analytics, makes a difference between truth de facto et necessary truth; he insists on the fact that: 1. The conclusion of a concluding syllogism might be true while the premises are not both true; in that case, the conclusion is not necessary. 2. The premises and the conclusion of a non-concluding might be all true; in that case, also, the conclusion is not necessary. Wittgenstein (1921) and Tarski (1944) are in the continuity of Aristotle perspective concerning truth and validity. For both authors, a sentence is true if what is expressed is in adequation with the fact that it is supposed to describe. Wittgenstein (1921) enlightened the distinction between non logical proposition, and logical proposition, that are the theorem of the formal system. He claims that it is clear that a proof in logic and a logical proof in mathematics are two things radically different, and that logical validity is essential, while general validity corresponding to a true generalized conditional is accidental. Tarski (1933, 1944) as for him, intend to build a semantic definition of truth both materially adequate et formally correct allowing to keep the interest of syntactic methods, relying on validity, and to articulate them with mathematics contents, in search of truth. The methodology of deductive science (Tarski, 1936) provides a very powerful method for exploration the relationship between truth and validity that are at the very heart of proof and proving in mathematics. In Quine (1950), this distinction is very clearly described and justified; it opens path for examination of questions related to axiomatic.

1.3. Some consequences for mathematics education

As we have shown in Durand-Guerrier (2003), all the forms of implication met in our inquiry are presents in learning and teaching mathematics, while this complexity is most often underestimated by teachers, leading to deep misunderstandings between teachers and students. As a first consequence, it is necessary to reconsider the *dogma* that in mathematics class, a statement is either true or false; mathematical activity requires the manipulation of contingent statement. A second consequence is that the common practice of implicit quantification of conditional statements, that is not shared by most students, hides the differences between open and close conditionals, that is crucial for understanding the deductive reasoning; the third one is that it is necessary, at least at tertiary level, to emphasize the distinction between truth in a theory, and a logical proof of such a truth. Last but not least, we have shown that taking in consideration semantics in interpreting students' answers plaid in favor of a relative continuity between mathematical logic, and ordinary logic, far from the radical difference often assumed in psychology or in mathematics education (Durand-Guerrier, 2005).

II. Mathematical reasoning between natural language and formal language

It is often considered, in philosophy, that mathematics is the kingdom for exactness and rigor, and that to be recognized as scientific, a discipline has to be formalized in a mathematical language. In this perspective, following Quine (1987), predicate calculus ought "to attest to conceptual clarity" (p.158). However, as an apparent paradox, many tertiary level's students feel strong difficulties in dealing with formal languages. More over, sometimes, the use of formal language prevents students from elaborating a correct complete proof, although they have recognized the relevant arguments allowing to conclude. We have particularly worked on two categories of difficulties: those related to the *interactions between negation and quantification* (Durand-Guerrier & Kilani, I., 2004), and those related *with multiple quantification*. We will discuss here the second point, entering by a discussion about the logical status of letters

II.1 About letters' status

As well in mathematics class as in textbooks, it is common that the logical status of letters in mathematical statement is instable. In order to overcome the ambiguity due to this instability, it seems that mathematics teachers use to lean more on mathematics knowledge of the context than on an explicit requirement to the resources of quantification (Durand-Guerrier & Arsac, 2003, 2005). This is particularly the case while dealing with statements involving two different quantifiers on the type "for all x, there exists y such as $F(x, y)$". More generally, several researches in mathematics education enlighten difficulties faced by students at tertiary level to master predicate calculus and, particularly, concerning this type of statement (for exemple: Selden & Selden, 1995, Dubinsky & Yiparaki, 2000). For teachers, the difficulty seems to arise from the following paradox: how is it possible to provide reliable means of control to allow a correct use of multi quantified statements, if these means rely on the mathematics knowledge that are to be taught?

The history of Calculus, and more specially the history of the concept of *limit*, rich in errors and controversies leading in particular to the concept of uniform convergence, attests that this difficulty is epistemologically consistent. It is nevertheless well known that a solution consisting in teaching first order logic as a prolegomena to advance mathematics teaching does not seem to be able to provide a relevant solution to these difficulties, those mainly because mathematical and logical are closely intertwined.

We will try to explain now how our experimental practice of this type of analysis relying on *Tarski's elementary model theory* or *Hintlkka's Game-Theoretical Semantics* leads us to more philosophical questions.

II.2 Foundation *versus* understanding

Research for logical foundation for mathematics is a main part of philosophy of mathematics all along the twentieth century. From our didactical point of view, the question becomes: how to ensure that students follow logical rules warrant of the validity of the reasoning? According with Wittgenstein (1969, §28), this would be impossible; that means that logic is unable to provide a foundation to mathematical activity. In our didactic perspective, a less ambitious aim consists in trying to understand mathematical activity. In coherence with Sinaceur (1991), the first author of this paper has shown that a model theoretic point of view provides a relevant methodology for analyzing students' activity related to proof and proving (Durand-Guerrier, 2005). We thought that this question of foundation *versus* understanding is addressed both to philosophy and didactic.

Theory of Didactical Situations in Mathematics (Brousseau, 1997) places at the very heart of the learning process, the evolution of a strategy in a situation in which dialogs play a fundamental role. To model the situation of *explicit validation*, Brousseau refers to Lorenzen (1967). However, it is clear that Lorenzen's point of view, essentially syntactic, is too poor to capture the richness of what happens during these dialogs: putting the focus on statements, it excludes mathematical objects, that are the core of mathematical activity. As for us, we consider that it is necessary to integrate objects while analyzing dialogs; indeed the actions with objects and the consecutive retroactions contribute significantly to the emergence of strategies. We have already shown that Tarski's elementary model theory offers relevant tools in this perspective (Durand-Guerrier, 2005) ; however, the dialogic aspects are not taken in account in this theory. Our hypothesis is that the game semantic theory developed by Hintikka, that associates Tarski's and Lorenzen's points of view, is a good candidate for developing a semantic point of view on dialogs and providing relevant tools for enlightening the dialectic between the construction of objects and the construction of theory in mathematical activity (Heinzmann, 2006).

II. 3 Game-Theoretical Semantics for modeling students' activity

In his current work, the second author of this paper explores the interest of the Game-Theoretical Semantics (Hintikka & Sandu, 1997) to analyze dialogs between students solving together a problem, and particularly during the situations of validation (Brousseau, 1997). According to Hintikka, Game-Theoretical Semantics is an alternative to the identification of dialogical logic with the "knowledge-acquiring activity" (Hintikka, 1983, p.39). A description of the method and an example of its use for analyzing a dialog can be found in Vernant (2007). Our own work leads to open a discussion on the adequacy of these theories with the effective mathematical activity. For example, Marion (2006, p. 11) says that « Lorenzen argues that his rules are abstracted from what he called our 'practical nonverbal activity' […] or our 'prelogical speech practice' […]. »). Far from identifying students' activity with the model used for analyzes, we have, according to the situation, chosen one or the other theory (Barrier, to appear). Even if the two theories are most often available, due to the correspondence described in Rahman and Tulenheimo (2006), the main interest of Game-Theoretical Semantics is to be able to integrate in a natural manner the experimental dimension of mathematics, characterized by a *va-et-vient* between mathematical objects and theory, that is more or less neglected in mathematics education in France, although it is now clearly mentioned in curriculum. This perspective appears to be really promising.

Conclusion

Our paper ought to be a contribution to the debate concerning philosophy and didactic, opening on the great interest of a fruitful dialog between didacticians and philosophers. Indeed, Philosophy offers a consistent epistemological reference for didactic analysis, and conversely, the experimental didactic study of mathematical activity in context may give some insights to philosophers about "what are mathematics really" and "how do we know about them".

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