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LOCAL THEORETICAL MODELS IN ALGEBRA LEARNING: A MEETING POINT IN MATHEMATICS EDUCATION

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Abstract

The need of interpreting unanticipated phenomena, arising from studies on the transition from arithmetic to algebra in the 80´s, gave rise to the long-term research program *Acquiring Algebraic Language*. The theoretical formulation in this program is characterized by its condition of local elaboration linked to specific phenomena under study, which makes it possible to probe the nature of such phenomena in a way that bordering disciplines meet face to face with mathematics education. Through an example about operating the unknown, the paper discusses how this multidisciplinary theoretical job brings together the work of specialists who have analyzed algebra language from the standpoint of linguistics, semiotics and pragmatics. Implications for algebra teaching practice, when working with this theoretical perspective, are also discussed.

Introduction

Mathematics education is a discipline located half way between the exact sciences and social and humanistic sciences. Moreover, its multi-disciplinary nature has made attempts at characterizing it an enormous challenge. Works devoted to meeting this challenge head on have contributed to defining the tasks, methods and theoretical foundations of mathematics education (see, for example: Freudenthal, 1973; Sierpinska & Kilpatrick, 1998; Lerman, Xu & Tsatsaroni, 2002; Boaler, 2002, among many others), yet in all cases there is an acknowledgment of the fact that the disciplinary boundaries it shares with converging areas of knowledge are very vague, particularly as regards to mathematics itself (see Goldin, 2003; Dörfler, 2003). This difficulty in determining disciplinary limits is usually aggravated in work aimed at interpreting observations made during empirical studies, especially if it is a matter of theoretically interpreting findings or results that were unanticipated (and perhaps not anticipatable) from the observation design stage. In many of the above-mentioned cases, one tends to turn to the theoretical frameworks derived from those other bordering disciplines, such as psychology, history, epistemology, semiotics or sociology, but the general frameworks of such disciplines do not always respond to the analysis needs of what is observed. This paper deals with just such phenomena, as reported by studies on the transition from arithmetic to algebraic thought in the 80s, and which led to a long-term research program that proposed development of theoretical elements that would make it possible to fine tune the very analysis of such phenomena.

One of the special features of the theoretical formulation in the research program *La Adquisición del Lenguaje Algebraico* (*Acquiring Algebraic Language*) [1] undertaken in the later 80s, is that although the point of departure is a general notion, that of a *Mathematical Sign System*, its very nature of local elaboration makes it possible to delve deeply and, hence, to generate new knowledge on the matter under study. In local treatment under the theoretical light in question, the perspective of *Local Theoretical Models* (developed by E. Filloy in the 90s) opens avenues among the multiple components that usually make the research problems up, instead of approaching them from a partial perspective. Indeed each local model contemplates the study of cognitive, formal mathematical competency, teaching and communications aspects. It is precisely this comprehensive approach that I would like to emphasize here, given that it raises the possibility of substantively contributing to highly focused research, within multiple disciplines and, what is better yet, because the scope of such contributions depends upon significant exchanges between specialists and communities related to those disciplinary fields. The paper goes on to present one of the didactic phenomena that motivated this theoretical development, to then illustrate its usage by way of a few examples. While the final sections of the paper discuss the implications for disciplinary confluences, from a theoretical standpoint, and the possible repercussions in the area of mathematics teaching, from a practical standpoint.

The Polysemy of X: Manifestation of the First *Didactic Cut*

"From the outside-in" type of theoretical interpretations have been explored in the Mexican project *Acquiring Algebraic Language*, in which the phenomenon of the *polysemy of x* was reported on at a time when the general theoretical frameworks of mathematics education failed to provide sufficient elements for it to be described, let alone explained. The *polysemy of X* consists of the spontaneous reading that teenagers tend to engage in with equations of the type: $X + X/4 = 6 + X/4$, by saying that "this X (the first term on the left hand side) is equal to 6 and that these two (the Xs that appear in the terms X/4 on both sides) can have any value, but that both must have the same value" (Filloy & Rojano, 1989). The item intended to make it clear in clinical interviews situations that one is in the presence of "term to term equalization" resolution strategies, which make it possible to identify the identical terms (X/4) and to infer the equality between the remaining terms (X and 6). At that time, as has already been mentioned, not only were there no theoretical elements to analyze the children's favorite response (*polysemics*), but even the appropriate vocabulary needed to refer to it failed to exist. Thus terms were borrowed from semiotics in order to refer to the fact that the first step in the equalization ("this X is 6") comes from a reading made in the semantic field, to which restricted equations belong, in which case the X is an unknown, while the second step of the equalization ("these two can have any value") is derived from a reading made in the semantic field of tautological equations or algebraic equivalences and, in such case, the X is a general number. Hence the expression the *polysemy of X*, since it is a matter of allocating meanings derived from different semantic fields of algebraic language to one and the same symbol within one single algebraic formulation.

A specialized vocabulary would be needed just to engage in a mere description of the phenomenon, but in the process of delving into the origin of polysemous readings other needs arose, such as speaking to the cognitive nature inherent in tending to make this type of reading in algebra, as well as the possible impact of prior learning (in the field of pre-algebra) and the role that could be played by this type of interpretation in conceptualizations the likes of algebraic equality, mathematical unknowns and in learning formal methods for solving linear equations. A clinical interview undertaken with 12 and 13 year old students, who had not been taught how to use algebraic methods to solve such equations and who demonstrated that they engaged in a polysemous reading of the equations, provided data that made it possible to answer the research problem's underlying questions: Do the characteristics of the students' spontaneous solution of linear equations, in which operating the element represented is necessary (for instance, in which the unknown appears on both sides of the equation), attest to the presence of a didactic cut point in the transition from arithmetic thought to algebraic thought? (Filloy & Rojano, 1989). In the case of the different modes of linear equations used in the study, the answer was yes; in the particular case of the equations that led to polysemous readings, the answer is yes as well, and the arguments based on detailed descriptions of those readings can be found in the reports published for several years starting in 1984 (Filloy & Rojano 1984, 1985a, 1985b, 1991; Filloy, 1991; Rojano, 1986, 1988).

Amongst other things, the *polysemy of X* showed the need for an in-depth re-conceptualization of equality in mathematics that went well beyond its arithmetic meaning (as C. Kieran had already pointed out (Kieran, 1980)), thus in turn enabling conceptualization and manipulation of unknowns in the syntactic field of algebra. Attempts at drawing students' focus away from their polysemous reading, by restating to such students formulation word problems that corresponded to equations of the X + 5 = X + X type led nowhere, because although they were well willing to accept that all occurrences of X had the same referent within the context of the problem, once the question at the syntactic level was asked "what is the value of X?", the response was once again "this X (the first on the right side) is 5 and these two (the first on the left hand side and the second on the right) can have any value". The persistent manner in which they focused on certain ways of reading algebraic expressions led to discussions of intermediate language strata between arithmetic and algebra, as well as of cognitive tendencies during periods of transition leading to algebraic thought. E. Filloy expressed the need to be very specific when dealing with phenomena of this nature, in which the general theoretical frameworks prevented deep analysis (Filloy, 1999), and proposed development of local models. Said theoretical development has over time made it possible to study the evolution toward algebraic thought in teenagers from a perspective that considers algebra as a language and in which competent usage thereof is preceded by abstraction processes found in language strata that come before that of the *Mathematical Sign System of Algebra*. Then a brief reference is made to this theoretical proposal, to then go on to illustrate its application in the case of a *second didactic cut*: when there is a need to operate something unknown, and when said unknown is represented in terms of another unknown quantity.

Local Theoretical Models and Mathematical Sign Systems

E. Filloy defines *Local Theoretical Models* (LTMs) according to the following four characteristics: (for an extensive presentation, see Kieran & Filloy, 1989; Filloy et al, in process): 1) an LTM consists of a set of assumptions about a concept or system; 2) an LTM describes a type of object or system by attributing an internal structure to it, which when taken as reference will explain several of the object's or system's properties; 3) an LTM is considered an approximation, which is useful for certain purposes; 4) an LTM is often formulated and developed based on an analogy between the object or system that is described and another different object or system. The author goes on to mention a series of differences that exist between model and theory so as to make it very clear why he emphasizes the term *model* within the perspective he adopts.

In the foregoing light, symbolic algebra is considered a language and there exists the interest, as has been previously stated, in also studying the relationship between the latter language and prior or intermediate language levels, thus incorporating the notion of a *Mathematical Sign System* (MSS) in the broad sense of the term (Filloy, 1999, Puig, 1994). Consequently MSSs are sign systems in which a socially agreed possibility exists to generate *signic* functions. As a result, the definition has room enough for including cases in which functional relations (*signic*) have been established for use of didactic devices within a teaching situation and in which usage thereof may be intentionally temporary. In other words, also considered or included are sign systems or sign system strata produced by students in order to give meaning to what is presented to them within a teaching model, even when said systems are governed by correspondences that have not been socially established, but that are rather idiosyncratic.

The notion of MSS plays an essential role in (locally) defining the components that make up the LTM and that deal with different types of: 1) *teaching models*; 2) *models for cognitive processes*; 3) *formal competency models*, which simulates the performance of an ideal user's of a SMM; and 4) *communication models*, in order to describe the rules of communicative competency, text formation and decodification, and contextual and circumstantial disambiguation.

The Second Didactic Cut, Under the Light of LTMs

Simply put, a *didactic cut* in the transition from arithmetic to algebraic thought consists of the manifestations of students, who are facing tasks of an algebraic nature for the first time, as regards the need to build new meanings and new senses for arithmetic objects and operations, with the added special characteristic that such newly built meanings and senses necessarily presuppose a *break with* arithmetic. The study *Operación de la Incógnita* (*Operating the Unknown*)[2] provides evidence of this type of manifestation, at the time when students first face the task of solving equations in which an unknown must necessarily be operated, for instance equations of the AX + B = CX + D type (for a more detailed description, see Filloy & Rojano 1984, 1989). The *polysemy of X* described in preceding paragraphs is precisely one of those manifestations that we could call the *first didactic cut*. In this case, if algebraic equations are to be interpreted, new meanings for equality and unknowns must absolutely be built, and operations dealing with these mathematical objects must be endowed with new sense.

The analysis carried out in the study for the first cut led to surmising the presence of other breaks, in particular terms conjecture arose regarding the manifestation of *a second cut* at the time when students are first faced with unknowns and when the latter are represented in terms of another unknown. A corresponding algebraic task is the solving of systems that contain two linear equations with two unknowns, that is, systems of the type:

$$Ax + By = C$$
$$Dx + Ey = F$$

(A, B, C, D and F are known numbers, x & y are the unknowns), whose solution eventually requires to manipulate expressions like:

$$y = (C – Ax)/B$$

where one of the unknowns (y) has a representation in terms of the other unknown (x).

Expansion of the study *Operación de la Incógnita* (*Operating the Unknown*), so as to delve deeper into this new cut point, produced data that are recently being studied from the perspective of *Local Theoretical Models* (Filloy, Rojano & Solares, 2003, in press). In order to accomplish the foregoing, components of the specific LTM have been elaborated and new data have been compiled and analyzed (Solares, 2003, 2004). Said elaborations are briefly described below in order to have examples at hand that illustrate the potential inherent in this theoretical perspective, but primarily to that they can serve as a basis to, in the next section, re-touch upon the issue of disciplinary boundaries and of LTMs as a meeting point for different areas of knowledge and for the corresponding specialist and teacher communities.

The Formal Competencies Model

The *Formal Competencies Model* of a *Local Theoretical Model* belongs to the realm of formal mathematics and its formulation consists of defining a *Mathematical Sign System* that enables decodification of texts produced within a problem learning situation (Filloy, 1999, Solares, 2002a, 2002b). For the case at hand, that of operating an unknown when said unknown is represented in terms of another unknown, it is a matter of defining the MSS of symbolic algebra in such a way that it is possible to analyze student responses in terms of the different contexts and semantic fields of algebra, as well as through different language strata located between natural language or the language of arithmetic and that of algebra. The mathematical actions that are interesting to describe under said terms are *algebraic equalization* and *algebraic substitution*, simply because both are the basis for the two solution methods for equation systems (2 x 2). With these requirements in mind a description of the *Formal Competencies Model* encompasses three different perspectives, namely: from the perspective of *syntax*, from that of *semantics*, and that of *pragmatics* (Solares, 2002a). In describing the model's *syntactic* aspects, D. Kirshner's (Kirshner, 1987) description of *algebraic syntax* was used; while for the semantic description, J.P. Drouhard's analysis of the *significances of algebraic writings* (Drouhard, 1992) was used; and finally, for the *pragmatic* description (in process), the plan is to resort to (amongst other things) an historical analysis of the evolution of algebra's MSS (see Solares, 2002a).

The Syntactic Perspective

As in Kirshner, who uses *generative and transformational grammar* to generate simple algebraic expressions and carry out transformations upon them, in the *Formal Competencies Model*, competent users produce a version of *superficial forms* of algebraic expressions. In *transformational grammar*, the transformation of said expressions lies in the corresponding *deep forms*, which reveal the structure of the productions in terms of the operations that make them up and their hierarchy. Using the *generative and transformational grammar* of algebra's MSS, Kirshner's work has expanded to production of equations and systems of linear equations and to the transformations that lead to their solution (Solares, 2002a), thus obtaining a syntactic and structural analysis of the transformations as though they were carried out by a competent algebra user. That is to say, modeling the formal competency to work with two unknown quantities, one expressed in terms of the other (Fig. 1 depicts the formal competency model for this particular case, using the generative and transformational grammar symbology used by Kirshner).

Transformation of equalization:

$$\mathbf{X} = (\mathbf{Y}\ \mathbf{O_1}\ \omega)\ \mathbf{O_2}\ \theta \quad \wedge \quad \mathbf{X} = (\mathbf{Y}\ \mathbf{O_3}\ \xi)\ \mathbf{O_4}\ \tau,$$

$$\leftrightarrow\ (\mathbf{Y}\ \mathbf{O_1}\ \omega)\ \mathbf{O_2}\ \theta \;=\; (\mathbf{Y}\ \mathbf{O_3}\ \xi)\ \mathbf{O_4}\ \tau.$$

where $\mathbf{O_2}$ y $\mathbf{O_4}$ are additions or subtractions; $\mathbf{O_1}$ y $\mathbf{O_3}$ are multiplications or divisions; ω, θ, ξ y τ are real numbers; and ω y ξ are not simultaneously equal zero.

Transformation of algebraic substitution:

$$\mathbf{X} = (\mathbf{Y}\ \mathbf{O_1}\ \omega)\ \mathbf{O_2}\ \theta \quad \wedge \quad (\mathbf{X}\ \mathbf{O_3}\ \tau)\ \mathbf{O_4}\ (\mathbf{Y}\ \mathbf{O_5}\ \xi) = \zeta$$

$$\leftrightarrow\ (((\mathbf{Y}\ \mathbf{O_1}\ \omega)\ \mathbf{O_2}\ \theta)\ \mathbf{O_3}\ \tau)\ \mathbf{O_4}\ (\mathbf{Y}\ \mathbf{O_5}\ \xi) = \zeta.$$

where $\mathbf{O_2}$ y $\mathbf{O_4}$ are additions or subtractions; $\mathbf{O_1}$, $\mathbf{O_3}$ y are multiplications or divisions; θ, ω, τ, ξ y ζ are real numbers; and ω, τ y ξ are not simultaneously equal zero.

Fig. 1

Semantic Perspective

This entails describing the significance of algebraic expressions and transformations that arise from their competent use, in other words, within the *formal competencies model*. In order to accomplish this, the notion of *significance* as developed by Drouhard and its four associated aspects are used: *reference*, which corresponds to the *algebraic evaluation function*; *sense*, which results from the *set of transformations* applicable to the expression; *interpretation*, which corresponds to the different readings made of the expression within the different *contexts* in which it may appear (such as the theory of numbers, analytical geometry, etc.); and *connotation*, which falls into *psychological significance* (which in turn depends on each individual) (Drouhard, 1992).

The aspects that Drouhard distinguishes as regards to the significance of algebraic expressions correspond to the significance given by a competent user. Whereas for purposes of the LTM presently under discussion, said distinction is extended to the analysis of pupil productions when tackling new algebraic problems, and said pupil productions are usually located at intermediate levels when representing and manipulating an unknown that is given in terms of another unknown.

Pragmatics perspective

The formal competencies model becomes complete when symbolic algebra is described from the *pragmatics* perspective and the research program includes analysis of the use of *substitution* in the historical evolution of algebraic language in trying to find out which strata of MSS incorporate *algebraic substitution* (Solares, 2002a). The research program includes analysis of ancient texts, such as the following:

- The *Arithmetic of Diophantus* (Book I), in which expanding upon L. Radford's analysis, one can say that Diophantus applied the equalization method to solve equation systems, as in problem 26 of Book I (Radford, 2001);
- The *Abbacus Books*, particularly the book by *Fibonacci* (Boncompagni, 1857), as well as previous analysis of these books (Filloy & Rojano, 1984). The language used in the *Abbacus Books* can be allocated at an intermediate stratum of the *algebraic system of signs*;
- *J. De Nemore's De Numeris Datis* (Hughes, 1981), which is a treatise on systems of quadratic equations written in a pre-symbolic stratum, but in which a type of

substitution is used, in fact calculus' *algorithm substitution* is used to solve the equation systems;

- *Viète's Analytical Art* (Witmer, 1983), which marks the birth of symbolic algebra;
- *Stevin's Arithmetic* (Stevin, 1634), in which, according to Paradís and Malet, the first formal algebraic substitution is carried out (Paradís & Malet, 1989).

The Model for Cognitive Processes

In the study *Operating the Unknown* [2], which deals with the *first didactical cut*, a series of cognitive tendencies were identified and characterized, and they have since served to analyze student responses to their first point of contact with operating unknowns. The following are some of the cognitive tendencies: (1) conferring intermediate senses; (2) returning to more concrete situations upon occurrence of an analysis situation; (3) focusing on readings made in language strata that do not enable solving the problem situation; (4) the presence of inhibitory mechanisms; (5) the presence of obstructions arising from the influence of semantics on syntax, and vice-versa; (6) the need to confer senses to the networks of ever more abstract actions until they become operations (Filloy, 1991). By the way, the *polysemy of x* is a number 3 type tendency, because a polysemous reading of an unknown prevents students from recognizing that the equation is a restricted equality, and from searching for a single value for X.

Cognitive Process Model in the Second Cut

The entire set of cognitive tendency categories from the first study constitutes a model for LTM cognitive processes in the study of the *second cut* point, in which 12 students were interviewed. All of the students had been taught to solve linear equations, in which the unknown has to be operated, and its corresponding word problems. None of the students had yet been introduced to solving linear equation systems. The items list was divided in two sections: word problems and syntactic tasks. Figure 2 shows the syntactic items.

S.1	$x + 2 = 4$ $x + y = 8$	**S.6**	$14 + x = 37$ $4 - y = 28$	**S.11**	$3x + 4 = 22$ $4x + 2y = 34$	**S.16**	$y - 6 = 3x + 20$ $5y - 4x = 64$
S.2	$x + y = 10$ $x - y = 4$	**S.7**	$45 - x = 17$ $x + y = 41$	**S.12**	$3 \times (8 + x) = 6$ $2x + y = 23$	**S.17**	$3x + 8y = 84$ $8x + 3y = 59$
S.3	$x + y = 9$ $2x + 3y = 23$	**S.8**	$x + y = 60$ $3x = 171$	**S.13**	$2 \times (3 - x) = 6$ $4x + 3y = 12$	**S.18**	$4x - 3 = y$ $6x = y - 7$
S.4	$x + y = 12$ $5x - 6 = y$	**S.9**	$2 \times (x + 6) = 84$ $x + y = 104$	**S.14**	$4 \times (3 - x) = 4$ $x + y = 13$	**S.19**	$3x - 2 = y$ $5x = y + 8$
S.5	$x + 33 = 48$ $x + y = 73$	**S.10**	$4 \times (x - 8) = 72$ $x + y = 17$	**S.15**	$x - y = 1$ $x + y = 5$		

Fig. 2

As can be seen, although the first 14 items on the list have an increasing degree of complexity, all can be solved without requiring *algebraic substitution*, as *numerical substitution* alone is enough. As such in this section the students were asked to find the value of X and of Y, using their own means to solve the problems. The spontaneous strategies used by the students manifested cognitive tendencies the likes of (2) *returning to more concrete situations upon the occurrence of an analysis situation*, given that in all cases they resorted to a trial and error strategy. The following extracts, taken from the interviews of two girls, illustrate this tendency:

SL.3. $x + y = 9$ $2x + 3y = 23$	L writes: ***0 + 9*** ***1 + 8*** ***2 + 7*** ***3 + 6*** ***4 + 5*** ***5 + 4*** Crosses the last line and then points out line by line, starting at the first one. She stops at "***4 + 5***" and writes: ***4*** ***5*** ***8*** ***15*** ***23*** **L:** The numbers are ***4*** and ***5***. **Interviewer:** ***4*** and ***5***? **L:** Yes. First thing I did was to obtain the possible additions which gave ***9*** as a result and then, by "trial and error": ***2*** multiplied by ***1*** is ***2*** (points out "***1 + 8***") for checking if they lead to the correct amount.	*Observations: L interprets these two equations as "linked", that is, as equations in which "x" and "y" have the same value in both equations. L writes the different ways for obtaining 9 through the addition of two positive integers and then performs the operations upon the unknowns indicated in the second equation until she finds those values for the unknowns, with which she obtains 23.*

Fig. 3

These spontaneous readings and strategies can obstruct learning of general solution methods such as is demonstrated in the following case:

SMt.17. $4x - 3 = y$ $6x = y - 7$	Mt wants to obtain the value of '***y***'. So, she transforms the proposed system into: $4x - 3 = y$ $6x + 7 = y$ but instead of performing the equalization of the two expressions, she looks for the solution through the "trial and error" method using only the positive integers. Mt: In here (***4x - 3 = y***) says that four times '***x***' minus three equals '***y***'. And here (***6x + 7 = y***) says that six '***x***'… plus seven, equals '***y***'. This (***6x + 7 = y***) has to be bigger than this (***4x - 3 = y***).	*Observations: Mt is able to solve one-unknown equations, regardless of the numerical domains of the operated numbers or of the solutions or complexity of the equations' algebraic structure.* *Besides, she uses comparison in the case of equation systems derived from verbal problems in which both equations have the same unknown solved and the solutions are positive integers.*

Fig. 4

Other results obtained based on the interviews indicate that in the process of learning general methods to solve equation systems, amongst other things developing new significances for equality is needed. The foregoing becomes increasingly clear as the children progress along the list of items to be solved. In the section on Teaching Models, this idea is dealt with in greater detail.

Teaching Models for Solving Two-Unknown Equation Systems

From the analysis performed at the formal level, the following *didactical route* for introducing the general methods was adopted –coming from the previously acquired competencies for solving one-unknown linear equations: (1) reduction of the two-unknown and two-equation system to a one-unknown equation by applying comparison or substitution; (2) solution of the one-unknown equation applying the previously learned syntax; (3) substitution of the numerical value found in one of the two equations; and (4) solution of the equation through application of the previously learned syntax. Depending on whether the comparison or substitution is applied in step (3), this route leads to classical *equalization* and *substitution* methods for solution of equation systems (2 x 2).

EXAMPLE :

(i) $x - y = 1$ (ii) $x + y = 5$ **First Route** Reduction, from (i): $x - y = 1$, then $x = 1 + y$ Substitution in (ii): $(1 + y) + y = 5$ Solving for y: $1 + 2y = 5$, then $2y = 4$, therefore $y = 2$ Numerical Substitution in (i): $x - 2 = 1$ Solving for x: $x = 3$	**Second Route** Reduction from (i) and (ii): $x = 1 + y$ & $x = 5 - y$ Comparison of the two expressions for x: $1 + y = 5 - y$ Solving for y: $2y = 5 - 1$, then $2y = 4$, therefore $y = 2$ Numerical substitution in either (i) or (ii), let´s say in (ii): $x + 2 = 5$ Solving for x: $x = 5 - 2$, therefore $x = 3$

School Methods for Solving Linear Equation Systems

Linear system (2 x 2) solution methods by *equalization* and by *substitution* may be introduced by following the didactic path described previously and assume, for the first case, equalization of two chains of operations or two manners of calculating the value of one of the unknowns ($x = 1 + y$ & $x = 5 – y$, in the Second Route of the example above) and for the second, the need to substitute the representation of one of the unknowns in terms of the other in one of the equations of the system ($(1 + y) + y = 5$, in the First Route of the same example). There is a tendency to take for granted that the significance of that equalization, which is algebraic, is generated as of the significances attributed to the numerical equalization. There is likewise the tendency to assume that the algebraic substitution is an extension of the numerical substitution. Nonetheless results of the study on the *second cut* report that only students who are able to make sense of the actions that involve these solution methods (the transitivity of the equalization, among others) are in turn able to generate new *significances* for the algebraic equality, as required by operation of the unknown at this second level of representation (Filloy, Rojano & Solares, 2003).

The results referred to here, interpreted within the framework of the LTM prepared ex profeso for this study, made it possible to decipher the *significances* and the *senses* of the chains of actions involved in the school methods, as well as the obstacles in learning them, and all based on the *formal competencies* model and the model for *cognitive processes* (see the examples in section IV.2.1 of this papers, entitled *model for cognitive processes*). Hence the implications for the field of teaching are described as the mathematical, semiotics and cognitive perspectives, through the integrating axis of the *MSS* and with a level of detail that would have been very difficult to achieve from more general frameworks, even where they to come from the field of semiotics or psychology.

Models in Technology Learning Environments (The spreadsheet method)

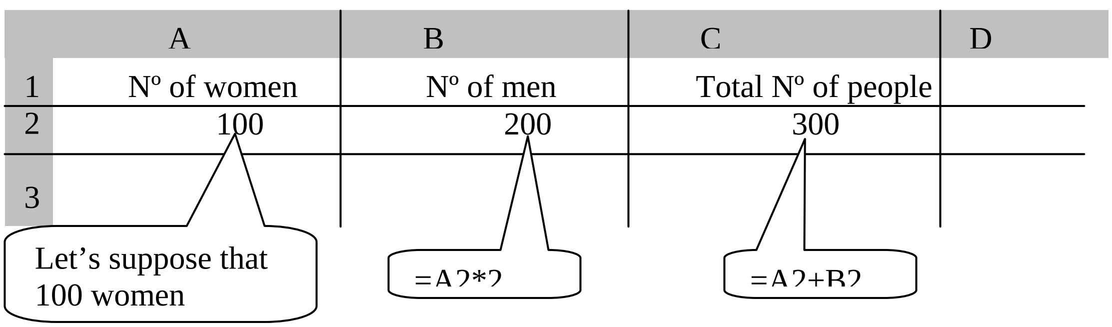
To date reference has only been made to the syntactic level of unknown operations at a second level of representation; in other words within a more abstract MSS stratum of algebra, than that of the stratum in which it is only represented by literal symbols. Nevertheless, in the field of teaching support from the referents of a word problem for introduction of manipulative algebra issues is absolutely fundamental. Even the *second cut* study incorporates a section with word problems involving two unknown quantities (Filloy, Rojano & Solares, in press). This part of the study includes the classic methods: the *Cartesian Method*, which assumes a translation from natural language into the algebraic code and the arithmetic method or *Method of Successive Analytic Inferences*. In addition, a third method is included, the *Method by Successive Analytic Explorations*, which is located at an intermediate stratum between the *MSSs* corresponding to the latter two (see Filloy & Rubio, 1993). The latter begins by assuming a numerical value (arbitrary) for an unknown and the analysis process of the problem formulation is carried out in terms of numerical relations without having to deal from the beginning with incorporating the unknown into the analysis. Thus the relationship between the data and the unknowns becomes explicit, eventually leading to writing such relationships in algebra's *MSS*.

The results that arose in this part of the study, as well as its theoretical perspective of the corresponding LTM, represent inputs that can be taken advantage of by revisiting the data obtained in the Anglo-Mexican *Spreadsheets Algebra Project* [3], which was undertaken in the 90s. The project showed that it is feasible for pre-algebra students aged 9 through 12 to solve, with the aid of a spreadsheet, word problems involving 2, 3 and 4 unknown quantities. One explanation given for this fact is that said computer environment helps students in the analysis phase and in translating the word problem into a language that is similar to that of algebra, since it enables them to bear in mind the symbol referents in the problem during the analysis phase. A second reason is that students do not have to master symbolic-algebraic manipulation in order to solve linear equation systems, because in that phase the solution is numerical and is obtained by means of automated calculation procedures (Rojano & Sutherland, 1991, Rojano & Sutherland 2001; Sutherland & Rojano, 1993). Then an example is provided in order to illustrate the spreadsheets method used in the Anglo-Mexican project; the algebraic method is described also, so as to show the syntactic mastery requirements for its application.

THE PARTY PROBLEM (a simple case):

420 people attended a cocktail party; the number of men was twice the number of women. How many women and how many men went to the party?

Algebraic Method:
If
x= # of women
y= # of men
Then y = 2x
And x + y = 420
After solving this system of equations, it is found that: x = 140 and y = 280, which is the solution to the problem.
Spreadsheet Method:
Identify the unknown quantity (or quantities) as well as the problem data. Suppose that, that which is unknown, is known, and allocate an arbitrary value to one of the unknown quantities, for example the number of women. This number is then introduced into one of the cells. In the neighbouring cells, introduce the corresponding formulas for the number of men and the total amount of people who attended, as shown in the following diagram:



Note that these formulae shall include the name of the cell of one of the unknown quantities.

The presupposed value is then changed until the number in the cell relating to the total number of people corresponds to the problem data (420). The following diagram shows the moment in which this value is obtained and, as a result, the correct values for unknown quantities.

	A	**B**	**C**	**D**
1	Nº of women	Nº of men	Total Nº of people	
2	140	280	420	
3				
4				

Fig. 6

The spreadsheet method can help in the analysis of the problem's text by recording the steps of this analysis in a system of representations, in which natural language (column labels) is used along with numerical language and an algebra-like symbolic language. Consequently, the analysis process, which consists of clearly stating the relationship between elements of the problem (data and unknown quantities), uses all of these languages:

- Natural language allows the presence of referents which provide the context of the problem;
- Formulas allow relationships between data and unknown quantities to be expressed and, more importantly, allow functional relationships between unknown quantities to be expressed;
- The supposition of a specific value for one of the unknown quantities allows the analysis and symbolization process to be undertaken, through the use of a known number instead of an unknown quantity.
- The numerical variation of the assumed value for one of the unknown quantities incorporates one of the intuitive methods most frequently used by students, the method of trial and refinement.

In the case of the PARTY PROBLEM, the spreadsheet method is illustrated using a simple case. However, this method can be used for the solution of problems of different levels of complexity. For example, problems where the relationships between unknown quantities are more complex or problems which include a larger number of unknown quantities, as shown in the following example:

SWEETS PROBLEM:

500 sweets are to be shared between three groups of children. The second group receives 20 more sweets than the first group and the third group receives three times as many sweets as the second group. How many sweets does each group receive?

Algebraic method:

If

x= # of sweets for the first group

y= # of sweets for the second group

z= # of sweets for the third group

then

y=x+20

z=3y

x+y+z=500

Solution:

x=84, y=104, z=312

Vary the number in cell A2 until 500 (specified in the problem) appears in cell D2

	A	B	C	D
1	Nº of candies First group	Nº of candies Second group	Nº of candies Third group	Total Nº of candies
2	30			
3				
4				

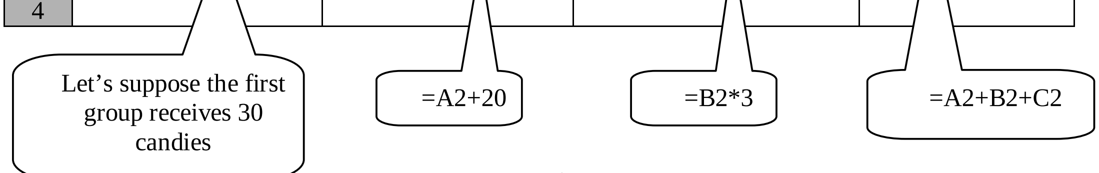


Fig. 7

Although the majority of the results reported in this project suggest that the *spreadsheets* method may make the tasks of solving problems that have more than one unknown accessible for very young students because it is not paradigmatic, to solve said method is nothing other than a didactic device used to place students on the road towards the *Cartesian Method*. And this situation raises questions, such as at what point can the two methods come together? Does the presence of the referents, which is very useful during the translation to spreadsheet symbolism phase, end by being an obstacle in developing the syntactic skills needed to solve the corresponding equation system? Or does the dependence upon a numerical means of solving the problem anchor students to the field of what N. Balacheff calls symbolic arithmetic? (Balacheff, 2001).

The foregoing questions have been broached by resorting to an analysis scheme proposed by (Puig & Cerdán 1990) in order to classify word problems as either arithmetic or algebraic, depending on whether the translation process leads to a chain of operations that only involves data or if said process leads to a chain of operations involving an unknown quantity. In other words, it depends on whether or not the process leads to an equation . The authors use two general methods as their analysis tools: *the method of analysis and synthesis* and the *Cartesian method*. By applying this scheme, the results of the Anglo-Mexican Project suggest that translations from natural language into the spreadsheets code, as undertaken by the students, has several characteristics in common with the translation into the algebraic code used in the *Cartesian method* (Rojano, 2002; Rojano & Sutherland, 2001). Nonetheless, regardless of whether these processes are of an algebraic nature or if they fall into the field of *symbolic arithmetic*, some researchers still consider them valid provided a full breaking away from arithmetic is not shown (Balacheff, 2001). As such the issue of things “algebraic” in children’s solution processes using spreadsheets can now be touched upon from the standpoint of *LTMs*, and particularly from the point of view of the components of teaching models and cognitive processes. At the end of the day, this is a matter of analyzing both the inter-relationships between the variety of MSS’ that come into play in both solution methods (Cartesian and Spreadsheets), as well as the cognitive tendencies that arise during their use.

The components of LTMs and Disciplinary Boundaries

With this walk through the history of the development of a research agenda on acquiring algebraic language, including one of its derivations to the application of the use of technology learning environments, the idea had been to show a particular manner wherein there is a point at which bordering disciplines meet face to face with mathematics education. The framework of that convergence is of a theoretical nature, in which one of the traits, that of local development (ex profeso for a series of phenomena that is presented in the transition towards algebraic thought), often puts at the limit the possibilities of such disciplines to advance our knowledge as regards the different aspects of the phenomena studied. In the example described here, the linguistics, semiotics and mathematical inputs used to prepare some of the LTM components were not incorporated in their raw state; helping hands were taken from the work of specialists who had broached the none too trivial task of analyzing the algebraic language from the standpoint of those particular disciplines. In this case, the boundary lines between said disciplinary fields and mathematical education are covered by that particular specialized work. Likewise, elaborations underway, such as that of the formal competencies model, from the point of view of the pragmatics of language, are based on prior work on the history of algebra that analyze the uses of algebraic language versions in ancient texts which correspond to stages prior to the emergence of symbolic algebra in the 16th C. Hence the *LTM* of the *second cut* may be considered a point at which linguistics, semiotics, pragmatics, history and algebra actually meet.

Algebra as a Language: Connecting Communities

The disciplinary confluence of Local Theoretical Models should not be seen as an exercise of extreme eclecticism or as puzzle that needs to be put together. The theoretical axis of consistency is represented by the notion of a Mathematical Sign System, which makes it possible to maintain the formal, historical, "in use" and school versions of mathematics (algebra, in our case) at the core of the research work. Whereas the idea of using this underlying notion arises from the view of treating algebra like a language, this view is shared by many researchers who have expressed their analysis, thoughts and perspectives, not only through formal scientific publications, but also in the close interaction possible in international symposia and seminars (see for example: Arzarello,1993; Bell,1996; Drouhard,1992; Filloy,1990; Kaput,1993; Kirshner,1989,1990,2001; Lins 2001 Mason,1989; Pimm,1995; Radford,2000). In particular, the Algebra Working Group held substantial and heated discussions for several years on the issue within the framework of the PME meetings (international and North American). Furthermore the seminar on semiotics and algebra held on Fridays at the Department of Mathematics Education at Cinvestav-Mexico, has incurred in new variants with the arrival of young researchers and their joining ongoing study of the issue. All of this prolonged and intensive interaction amongst algebra scholars has represented another important factor in enabling the theoretical conception expressed here to be a meeting point of neighboring fields of knowledge that extends well beyond a mere eclectic use of their theoretical elements.

Final Remarks: Theory and Practice

Focusing in on theoretical analysis in the *second cut point* also reveals another type of connection: the link between theory and practice. Detailed development of the *formal competencies* model component, as well as knowledge of cognitive tendencies that flourish in the form of *reading levels* or the interpretation of linear equations in different strata of algebra's MSS, all led to the didactic route used in the clinical interview for the respective study. On the one hand said route anticipates the difficulties faced by students in manipulating unknowns in the *second cut*, in view of the structural syntactic characteristics of linear systems and of developing the sense required to undertake their transformations; these two conditioning factors are revealed by the syntactic and semantic components of the formal competencies model. While on the other, this trajectory suggests linear system modalities which, if presented gradually, place students in the position of needing to advance conceptually in algebraic *equalization* and *substitution*. The foregoing demonstrates how theoretical work on the formal and cognitive components of LTMs becomes one of the basic inputs (directly usable) for formulating a *teaching model*, which is actually located in the field of practice.

The task of bridging the gap between theory and practice shall only be complete when the figure and role of teachers is incorporated into theoretical treatises. As regards the case at hand, that is to say LTMs for the *second cut*, the task will have been dealt with when a local communication model has been developed, a model that encompasses the text formation and decodification processes that are triggered during teacher-student interactions. That is precisely when the elements will be available for the teacher -just as Balacheff indicates- to play a role in enabling students to recognize the *cut point*, an experience that simply cannot be done away with. In other words, the teacher will be able to play the role of master of the rupture (Balacheff, 2001) in the algebra lesson.

Notes

[1] *Adquisición del Lenguaje Algebraico (Acquiring Algebraic Language)* is a research program, conducted by Eugenio Filloy and Teresa Rojano at the Centre of Research and Advanced Studies (Cinvestav) in Mexico, since 1980, which intends to probe the learning processes of algebra, when the latter is consider as a language.

[2] The “non-operation on the unknown”, which might seem to be related to the presymbolic character of ancient algebra texts (belonging to pre-Vietan stage) led to the formulation, in the field of mathematics education, of conjectures as to the presence of “didactic cuts” in the processes of transition from arithmetic to algebraic thought. *Operación de la Incógnita (Operating the Unknown)* is the clinical study that aimed to confirm such conjectures at the ontological level. Eugenio Filloy and Teresa Rojano undertook this study in the early 80´s at Cinvestav – Mexico.

[3] Project funded by the National Program for Mathematics Teachers Training, The National Council for Science and Technology (Grant No. 139-S9201) in Mexico, and The British Council. Part of this work derives from the ESRC funded project “The gap between arithmetic and algebraic thinking” (Grant No. R000232132) in England.

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