What is mathematical literacy?

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Freedom is the freedom to say that two plus two makes four.
If that is granted, all else follows. (George Orwell, *1984*)

My goal is to explore the relationship between

Mathematics and "*mathematical literacy"*

in a way that might encourage a more critical analysis of the many appealing, but often rather vague, claims made by the advocates of "mathematical literacy" (aka "functional mathematics"). My focus is on *school* mathematics in England – though there are good reasons for believing that similar observations apply elsewhere. My hope is that the analysis presented here may enable other countries to avoid some of the pitfalls that have characterized developments in England during the last 25 years, where a paradigm-shift from elementary mathematics to "numeracy" occurred after the publication of the *Cockcroft report* (Cockcroft, 1982).

Mathematics is more permanent than almost any other human cultural activity: school mathematics has its roots in methods often going back several thousand years. In contrast, concern about "mathematical literacy" (ML) is of relatively recent origin. These two strands in our analysis are sometimes difficult to disentangle; but the central points may be summarised as:

- mathematics is very different from (basic) "numeracy" or "mathematical literacy"
- "numeracy" and "mathematical literacy" are best seen as planned *by-products*, rather than as central goals, of effective mathematics education
- attempts to redesign school mathematics as in England during the period 1982-2007, to give it a tight initial focus on "numeracy", leaving more abstract ideas and methods to follow later, have had predictable negative consequences for *all* students; such attempts contradict both the character of elementary mathematics and the way human beings learn
- nevertheless, it may be helpful to consider what humane consequences these desirable practical outcomes could have for school mathematics; in particular, we end by trying to make a clear distinction between basic "numeracy" (or "quantitative literacy") and a broader interpretation of "mathematical literacy".

Mathematics and mathematics education inhabit different worlds. Mathematics is more precise, more "objective", and less subject to fashion than is mathematics education; yet its ways of working, its principles and insights are subtle, so need to be mediated in various ways before they can become an effective part of the world of mathematics education. On the other hand, if mathematics education and ML wish to benefit from their association in the public mind with the objective universe of "mathematics", they are obliged to represent that universe faithfully when mediating its subject matter for the pragmatic worlds of schools, teachers, and students; of politicians and bureaucrats; of curricula and examinations. In seeking improved approaches to elementary mathematics, we are not free to replace the content and objective character of mathematics by something more "user-friendly", but are obliged to respect the fundamental nature of the discipline.

In Peter Shaffer's play *Amadeus*, Mozart represents God (or here Mathematics, the eternal), while Salieri represents Mammon (for us "mathematical literacy" – the transient). Salieri is "flavour of the month"; but his influence is short-term and superficial. In contrast, Mozart represents the nearest that Man can come to God. In Shaffer's play Salieri understands this contrast perfectly well – and resents it bitterly!

"God needed Mozart to let himself into the world.
And Mozart needed me [Salieri] to get him worldly advancement."

Salieri consistently exploits his temporal influence to cut the heavenly Mozart down to size:

"What use, after all, is Man, if not to teach God his lessons?"

Salieri clearly understands the perfection of Mozart's art and the relative crassness of the world's judgment. After the first performance of *The Marriage of Figaro* he muses:

"Could one catch a realer moment? …
The disguises of opera had been invented for Mozart …
The final reconciliation melted sight.

Through my tears I saw the Emperor yawn."

Emperor [coolly]: *"Most ingenious Mozart. You are coming along nicely."*

Later Mozart, close to death, is reduced to repeating a childish tune and Salieri is jubilant:

"Reduce the man: reduce the God.
Behold my vow fulfilled.
The profoundest voice in the world reduced to a nursery rhyme."

Politicians (like Shaffer's Emperor) may misconstrue and trivialize *mathematics*; but mathematics educators should resist the temptation to take advantage of this distortion, and should never join with those who seek to replace this "universal heavenly music" by mere "tunes for the masses".

Mathematics teaching may be less effective than most of us would like; but we should hesitate before embracing the idea that school mathematics would be more effective on a large scale if the curriculum were to focus first on numeracy ("useful mathematics for all"), leaving more formal, more abstract mathematics to follow later for those whose interest survives this potentially misleading introduction. For example, basic arithmetic became accessible to all only when it adopted notation and written procedures (or algorithms) which combined the profound abstractions of "powers of 10", the index laws, and place value. And whilst one can program machines to do arithmetic unthinkingly, human beings are not machines: if they are to operate *mathematically* with simple numerical calculations (as human beings), they need to understand the abstract structure that underpins arithmetic.

The experience of committed teachers and small-scale projects in England reflects evidence from other countries: namely that it is perfectly possible to help many more students to achieve a useable mastery of elementary mathematics – but only if one is willing to interpret the goal of eventual “numeracy” *in a mathematical spirit*. (The “evidence” here – as in most educational judgments – cannot be easily summarized or aligned with any specific approach beyond the four immediately preceding italicised words. Examples range from (i) the approach adopted by certain complete education systems – such as Finland, Russia, Singapore, Hungary, evidenced by international comparisons such as TIMSS, PISA, or the Kassell project (Kassell, 2004); through (ii) medium sized projects – such as (MEP, 2008), or those using Singapore textbooks in other countries (e.g. (IFMA, 2008) or (SingaporeMath.com, 2008)); to (iii) the proven success over many years of committed teachers – as indicated by the subsequent performance of their graduates. If we take this evidence seriously, we might consider restricting the initial focus in school mathematics to truly basic material (integers, fractions, decimals, proportion, word problems, algebra and geometry), but teach this material in a way which encourages all students to use these ideas effectively and which at the same time prepares large numbers of students to move on to more serious mathematics when they need to. However, one should not be surprised if such a program turns out to look strangely like what good mathematics teaching has always been!

Since 1999 all English primary schools have begun each day with a “numeracy hour”, where teachers have concentrated on achieving mental fluency with such calculations as

$$15 \times 9 = ...$$

Rather than being based (i) on a didactical analysis of the mathematical abstractions that underlie arithmetic, (ii) on the development and piloting of associated instructional materials, and (iii) on deepening teachers’ “profound understanding of elementary arithmetic” (Ma, 1996), the English approach has been to encourage an unspecified “range of strategies”, with teachers being given no indication as to which calculational strategies are most important for subsequent mathematical development. After four years of intensive and highly focused effort, some observers were impressed when the 2003 TIMSS scores for pupils in Grade 4 showed a marked improvement over 1999 (TIMSS, 2008). However, there was no improvement at all in Grade 8! And there were clear signs of fragility even in the Grade 4 results: after more than 5 years of schooling (English pupils start school at age 4-5), 59% of English pupils managed to complete the calculation 15×9 correctly; yet pupils of the same age in countries which start school much later (e.g. Russia and the Far East), and where basic instruction does not focus so narrowly on a limited range of mental tasks, achieved success rates on the same task in excess of 80%, or even 90%!

There is no royal road to mathematics. Plausible-sounding “reform” rhetoric rarely translates easily into large-scale improvement in the classroom. So when faced with some new proposal, we have to exercise judgment to assess its likely efficacy.

In the last decade thousands of pages have been written on the general themes of “numeracy” and ML. So far, they remain largely aspirational. We may all agree that the school context is artificial, but that we should nevertheless work to ensure that students experience school mathematics as something “useable”. However evidence that a major shift of emphasis *at school level* of the kind proposed by some advocates of ML would lead to widespread improvement is at best mixed. In the UK the evidence is mainly negative: there has been a rise in average scores on *national* tests, but stagnation or decline on standardized international assessments; and there has been a marked decline in the observed competence of those entering universities at age 18 to study numerate disciplines.

In the Netherlands, where the shift has been much more carefully thought through, we are told that there have been some positive gains for "typical" students; but universities report worryingly negative consequences for those needing to pursue numerate disciplines beyond age 18.

Whenever "reform" in mathematics education makes common cause with political events (such as Sputnik, or an official inquiry such as (Cockcroft, 1982), or the TIMSS and PISA results), the conditions are set for some new magic fix to emerge ("new math" – see e.g. (Kline, 1973), or "problem-solving" – see e.g. (Gardiner, 1996), or "technology" – see e.g. (Gardiner, 1988), or "discrete math" – see e.g. (Gardiner, 1991), or "discovery learning", or "back to basics", or "constructivism", or "functional mathematics" (Smith, 2004)), which may then be welcomed without subjecting it to the obvious critical analysis. At such times we need to beware lest Snake-oil salesmen flourish, and those with honest products to sell, or valid criticisms to make, get sidelined.

The key to effective mathematics teaching is elusive. So when "mathematical literacy" and numeracy are presented as though they were *alternatives* to traditional school mathematics, rather than *by-products* of effective instruction (like "literacy" or "maturity"), there is a danger (i) that politicians and employers may seize upon the idea that there is a pragmatic-sounding alternative to "difficult" mathematics, (ii) that bureaucrats may imagine that focusing on numeracy from the outset might deliver what they see as the required (utilitarian!) end-product more directly and more cheaply; and (iii) that some educationists may see this paradigm shift as an opportunity to further undermine the idea of mathematics as the archetypal "objective" discipline. Such fears may be unwarranted; but as the examples given later show, it is important

- to look beneath the plausible-sounding surface to see whether the claims made for ML, and its siblings "numeracy" and "quantitative literacy" (QL), make sense *at school level*
- to ask whether there is a danger that, if applied unthinkingly *at school level*, the new emphasis on ML may become the latest in a series of bandwagons whose negative effects eventually outweigh any benefits
- to start work towards a possibly more useful interpretation of "mathematical literacy".

As long as success in teaching mathematics to a mass audience remains elusive, we must continue to explore alternative ways of teaching the subject. However, experience suggests that there is no magic bullet, and that "success" may elude us because mathematics and mathematics teaching are simply *hard*! If this is true, then any improvement may require painstaking didactical analysis, followed by design, piloting and planned implementation of *modest* changes to current practice, rather than some brand new paradigm. So we should perhaps hesitate before embracing the latest substitute for traditional school mathematics, and be prepared to return to the mundane world of identifying central principles, and to the hard graft of devising and testing incremental improvements. In particular, we should avoid being carried along on a flood of rhetoric - of which the following is one of many examples (Steen, 2001):

> "*Unlike mathematics, which is primarily about a Platonic realm of abstract structures, numeracy is often anchored in data derived from and attached to, the empirical world [and] does not so much lead upward in an ascending pursuit of abstraction as it moves outward toward an ever richer engagement with life's diverse contexts and situations.*"

Rather we should heed the warning of Hyman Bass (quoted in (Steen, 2004)), whose words describe with uncanny accuracy what happened in England in the late 1980s and 1990s:

"*the main danger … is the impulse to convert a major part of the curriculum to this form of instruction. The resulting loss of learning of general (abstract) principles may then deprive the learner of the foundation necessary for recognizing how the same mathematics witnessed in one context in fact applies to many others.*"

The origin of numeracy

The concept of numeracy emerged in the *Crowther report* (Crowther, 1959), where "numerate" is defined as

"*a word to represent the mirror image of literacy … an understanding of the scientific approach to the study of phenomena - observation, hypothesis, experiment, verification [- and] the need in the modern world to think quantitatively.*
Statistical ignorance and statistical fallacies are quite as widespread and quite as dangerous as the logical fallacies which come under the heading of illiteracy."

Note that the original meaning of "numeracy" was much broader than its more recent derivative. (The inclusion of "statistical ignorance" in this definition is interesting, but its intended meaning was probably rather basic, since at that time few mathematics teachers would have been in a position to teach elementary statistics.)

The claim that statistical ideas should be an integral part of basic numeracy re-emerged in the *Cockcroft report* (Cockcroft, 1982), where the topic was included as part of the *Foundation list* of material for *all* students (para 458), despite the admission hidden away in para 774 that "surprisingly few of the submissions which we have received have made direct reference to the teaching of statistics"! In recent years advocates of "quantitative literacy" have become even bolder in appealing to the evident importance of statistical examples.

It would indeed be wonderful if large numbers of students could somehow move beyond calculating with ordinary measures and learn that everyday "variability" can often be usefully analysed; but for that they would need to distinguish between statistical statements (about *populations*) and deterministic predictions (about individuals). Unfortunately probability and statistics are subtle disciplines, which have fooled all of us more often than we would care to admit. (For example, if we think about the *Monty Hall* problem in terms of a single event, there seems to be no reason to "switch". Only when we manage to think statistically in terms of "imagined repeated trials" does the paradox begin to resolve itself – though the initial gut-feeling can be strangely persistent.) So it may be optimistic to anticipate widespread competence in handling such material. Whether or not this pessimism is justified, we clearly need to concentrate in the first instance on achieving something much more basic.

For example, the simplest possible "statistical measure" of a population occurs whenever two quantities are directly proportional (i.e. vary *in a fixed ratio*). Hence our consistent failure to teach large numbers of students to handle simple problems involving ratios and percentages underlines the extent of the challenge we face. Moreover, the goal of achieving a robust everyday "statistical literacy" is complicated by the fact that statistical data as generally presented by politicians, in advertisements, or in the press, is often slanted to *persuade*, to *mislead*, or to *impress*, rather than to inform – with the result that even the most quantitatively sophisticated observer is likely to have trouble assessing the truth of what is claimed.

Data Handling is now one of the three content strands in the English National Curriculum; and after 20 years of repeated refinement, the structure, wording and interpretation of this strand still reveal the weakness of the "didactical analysis" on which it was based. Easy work on relative frequency and interpreting data certainly warrant attention within school mathematics; but devoting 20-30% of mathematics teaching time to uncomprehended cook-book statistics, as is common in English high schools, has done much harm. In particular:
(i) *it has reduced the time spent on more important material* (on which an understanding of basic statistical techniques depends);
(ii) it has had little impact on the level of "statistical ignorance" in the general population; and
(iii) it may well have contributed to a decline in interest in statistics among mathematics undergraduates (HESA, 2008).

Yet the claim that data handling is "more important" for ordinary students than some of the traditional core techniques which it inevitably displaces remains impervious to criticism. For example, when a recent government report (Smith, 2004) – written by a former President of the Royal Statistical Society – recommended unambiguously

> "*an immediate review of the future role and positioning of Statistics and Data Handling within the overall curriculum … informed by a recognition of the need to restore more time to the mathematics curriculum for the reinforcement of core skills, such as fluency in algebra …*",

all that happened was that officials and vested interests simply closed ranks. The subsequent "review" contract was awarded – without a tendering process – to the body which had been responsible for advising on the original curriculum! Instead of restoring more time for the reinforcement of core skills, the review took for granted the central importance of data-handling in the curriculum, and concentrated on developing materials to exemplify a "new problem-solving paradigm".

The fabulous statistics continued to pour out of the telescreen.
(George Orwell, *1984*)

We shall see evidence that the level of attainment in the UK – both in basic technique and in the ability to use the simplest mathematics – is currently so low as to make any concern about "statistical ignorance" largely irrelevant! Hence, rather than obliging schools to teach "statistics" before the necessary mathematical and scientific groundwork has been laid, we should concentrate at first on achieving a higher level of basic numerical competence so that school-leavers might be better able to see when plausible-sounding arguments, or numerical information, were being used in a misleading way.

Compared with many current advocates of QL, the *Cockcroft report* (Cockcroft, 1982) set its initial sights at a more realistic level, by starting out from the very basic notion that:

Numerate = "*able to perform basic arithmetic operations*".

However, it then introduced (para 39) two hostages-to-fortune for which we are still paying the ransom:

> "***the word 'numerate' [implies] the possession of two attributes.***
> *The first is an 'at-homeness' with numbers and an ability to make use of mathematical*

skills which enables an individual to cope with the practical demands of his everyday life. The second is an ability to have some appreciation and understanding of information which is presented in mathematical terms, for instance in graphs, charts or tables …"

For 20 years the profession in England has "talked the talk" of "at-homeness with numbers", without ever sorting out what was needed to "walk the walk". We have lost sight of the extent to which an appreciation and understanding of information presented in mathematical form *presupposes* an "at-homeness" with, and a mastery of, the relevant mathematical language **and procedures**. Instead the idea of "at-homeness" with numbers was presented as though it was an *alternative to*, rather than a *consequence of*, technical mastery and procedural fluency. As a result documents throughout the period 1979-1998 – from educational experts, from the official curriculum authority QCA, from the public examination bodies, and from Her Majesty's Inspectors (HMI) – repeatedly reinforced the message that:

- tables and standard written algorithms were deemed to be optional
- simple word problems and multi-step exercises could be neglected
- decimal arithmetic was delegated to the calculator and fractions were seen as outmoded
- euclidean geometry, proportion and algebra were viewed as beyond most pupils, and
- the resulting freed curriculum time was absorbed by such novelties as "data-handling".

Some consequences of the focus on numeracy

The *Cockcroft report* came at a time when England was struggling with the need to provide for "the bottom half" – a group which had traditionally been given a very raw deal. The report was unashamedly *utilitarian*, focusing on (school) mathematics *from the point of view of the needs of employment and adult life generally*. It was also supposed to consider the needs of higher education, but its attempt to address both issues at once foundered on its recommendation that the curriculum be designed *from the bottom up* (based on a "Foundation list"). That is, it suggested that school mathematics can be conceived in terms of a single curriculum "ladder" up which all students climb, at different speeds and to different heights, with pragmatic "*numeracy-for-all*" first, followed later by "*mathematics-for-those-who-insist*". The examples below indicate the impact this well-intentioned, but thoroughly misguided, utilitarian philosophy had on the teaching of basic technique: teachers and examiners could no longer sustain the expectation that large numbers of pupils should be expected to master techniques that are routine in other countries. Later we shall present evidence that *the ability to apply* simple mathematics has also suffered. This should serve as a lasting warning to us all.

TIMSS-R (1999: age 13-14)	**Success rates:**	**International average**	**England**
• **7003 – 4028** ***A. 2035 B. 2975 C. 3005 D. 3925***		**74%**	**51%**

- **4.722 – 1.935 = ??**

  A. ***<u>2.787</u>*** ***B. 2.797 C. 2.887 D. 2.897***

  77% **<u>58%</u>**

- **0.003) $\overline{1\,5\,.\,4\,5}$**

  A. 0.515 B. 5.15 C. 51.5 D. 515 E. ***<u>5150</u>***

  39% **<u>16%</u>**

- **$(^{6}/_{55}) \div (^{3}/_{25})$ = ??**

  23% **<u>4%</u>**

- **Find the value of y if $12y - 10 = 6y + 32$.**

  44% **<u>26%</u>**

Concern about the quality of those graduating from high school is not new. But the level of concern in recent years has been unprecedented. In 1995, after 15 years in which “reformers” had advocated a string of untested dogmas (for example, (i) that preoccupation with “mastery” of traditional content should give way to an emphasis on “understanding”; (ii) that there should be a shift in emphasis from “product” to “process”; (iii) that the availability of calculators replaced the need for everyone to “learn their tables”; (iv) that decimal calculator outputs could now replace the need to master the arithmetic of fractions; (v) that euclidean geometry should finally be laid to rest with the ancient Greeks; (vi) that serious algebra should be re-classified as something which consenting adult enthusiasts might be allowed to engage in in private; and (vii) that mathematics classrooms should be guided by “discovery”, by *children’s own* “reasons”, and by “investigation”), mathematicians in the UK finally lost patience – declaring (LMS, 1995) that:

> *“Mathematics, science and engineering departments appear unanimous in their perception of a* ***qualitative*** *change in the mathematical preparedness of incoming students.* Their criticisms … concentrate on three main areas.
>
> **4A** Students enrolling on courses making heavy mathematical demands are hampered by a serious lack of *essential technical facility* – [lacking] fluency and reliability in numerical and algebraic manipulation.
>
> **4B** There is a marked decline in students’ analytical powers when faced with simple two-step or multi-step problems.
>
> **4C** Most students entering higher education no longer understand that mathematics is a precise discipline in which exact, reliable calculation, logical exposition and proof play essential roles”

When this report appeared, government at first tried to brush the evidence aside. But three weeks later the TIMSS results were published, and officials were forced to sit up and take notice. The resulting *National Numeracy Strategy* (for England) was an attempt to “plug the leak” at primary level.

But the central problem has never been faced. In England, as in many western democracies, mathematics and other “hard” subjects (including physical sciences and languages) are in deep trouble. The pool of students is simply disappearing: despite the growth in the total number of high school graduates, in 2003 the number who had studied serious mathematics was around two-thirds of the corresponding number in 1989.

Year	Total number of "academic" exams taken at age 18	Total Maths	% Maths
1989	(Taken as) "100"	"100"	"100"
	↓	↓	↓
2003	**115**	**66**	**57**

In the same period, the *number* of undergraduate math-majors in England remained almost constant, though quality declined markedly. That there was no significant *decline* in numbers was largely due to the fact that funding to universities is based on recruitment *and retention*. Universities were faced with a stark choice: *adapt* (by moving the goalposts in order to pass those who should fail), or die. The stronger departments expanded to increase their income; but many mathematics (and science) courses and departments were closed.

The situation in many other countries is worse. In most western countries numbers staying on at school and proceeding to higher education have increased markedly; yet the quantity and quality of those specializing in mathematics at university have slumped. The reasons may be partly social; but mathematics education has not done enough to turn the tide.

The situation is serious. And when things get sufficiently bad, it becomes more tempting than ever to believe in the latest "magic fix", and to throw out what is left of the mathematical "baby" along with the bathwater. Yet even those who are strongly committed to the idea that high school graduates should be able to use elementary mathematics and make sense of simple quantitative information may find that the thousands of pages devoted to reports on "mathematical literacy" and "quantitative literacy" make depressing reading. Most of those who write on such subjects appear not to recognize

- that mathematics and mathematics teaching are simply *hard*
- that there is no "cheap alternative" to facing the fact that abstraction is a crucial part of elementary mathematics - almost from the outset
- that countries like England have already tried versions of what is now being proposed elsewhere, *and have paid the price.*

The current abysmal levels of achievement indicate the need for hard work and incremental improvement rather than the launch of yet another bandwagon.

My own attempts to pin down "mathematical literacy" – where this is advocated as an alternative to traditional elementary mathematics – have called to mind nothing so much as Lewis Carroll's classic parable *The hunting of the Snark.* Mathematical literacy as an appealing alternative to the hard grind of traditional school mathematics, like most other brands of educational "Snake-Oil", would seem to be a fiction, or "Snark". And those who sacrifice school mathematics to such a fiction may eventually be obliged to admit that the "Snark is a Boojum"! So I hope I will be excused for sharing extracts from

The hunting of the Snark
An agony in Eight Fits

by Lewis Carroll

"Just the place for a Snark!" the Bellman cried,
As he landed his crew with care;
Supporting each man on the top of the tide
By a finger entwined in his hair.

"Just the place for a Snark! I have said it twice:
That alone should encourage the crew.
Just the place for a Snark! I have said it thrice:
What I tell you three times is true."

He had bought a large map representing the sea,
Without the least vestige of land:
And the crew were much pleased when they found it to be
A map they could all understand.

"What's the good of Mercator's North Poles and Equators,
Tropics, Zones and Meridian Lines?"
So the Bellman would cry: and the crew would reply
"They are merely conventional signs!"

"Other maps are such shapes, with their islands and capes!
But we've got our brave captain to thank"
(So the crew would protest) "that he's bought us the best -----
A perfect and absolute blank!"

This was charming no doubt: but they shortly found out
That the Captain they trusted so well
Had only one notion for crossing the ocean,
And that was to tingle his bell.

But the central moral is that, if one sets out in pursuit of an ill-defined quarry, the "Snark" that one is busy hunting may turn out to be a "Boojum" - and with horrible consequences!

"It's a Snark!" was the sound that first came to their ears,
And seemed almost too good to be true.
Then followed a torrent of laughter and cheers:
Then the ominous words "It's a Boo---"

Then silence. Some fancied they heard in the air
A weary and wandering sigh
That sounded like "---jum!" but the others declare
It was only a breeze that went by.

In the midst of the word he was trying to say
In the midst of his laughter and glee,
He had softly and suddenly vanished away ----
For the Snark *was* a Boojum, you see.

In mathematics education as in life, mistakes are unavoidable. But it is time we learned *openly* from these mistakes. So let us hope that in future, when a "reform" promises much and delivers little, it will no longer be allowed to "softly and suddenly vanish away", but will be openly analyzed in a way that might contribute to cumulative professional wisdom.

> *Nothing is nobler than this endless struggle between the truth of today and the truth of yesterday.* (George Sarton)

> *Failure is instructive. The person who really thinks learns quite as much from his failures as from his successes.* (John Dewey)

Towards a conception of "mathematical literacy"

In the space that remains I shall move tentatively towards what I hope may be a more useful interpretation of "mathematical literacy". But first we have to answer the question as to whether "numeracy" and "mathematical literacy" are essentially *adult* competencies (like "maturity"), or whether "numeracy" and "mathematical literacy" are things one can genuinely teach and assess at school level.

"Maturity" is highly desirable; but no-one would claim that it should be "taught" or assessed at school level: it is rather an elusive *by-product* of an extended nurturing process, which includes detailed attention to a variety of specific disciplines and activities. Similarly *numeracy* and *mathematical literacy* would seem to be desirable by-products of school mathematics. If this analysis is correct, then it may make more sense to interpret "numeracy", or "quantitative literacy", as a basic

willingness to engage effectively
with quantitative information
in simple settings.

We could then preserve the term "mathematical literacy" to denote

a more subtle, long-term aspiration involving
simple insights into the nature of elementary mathematics and its applicability.

In some countries it would then be necessary to insist that, like any other "long-term aspiration", this kind of "mathematical literacy" – though highly desirable – cannot be measured using centrally imposed tests, and would probably to be trivialized if one tried.

Before we proceed on this basis we should admit that our modest interpretation contrasts starkly with "*mathematics literacy*" as defined by PISA (*Program for International Student Assessment*, OECD <www.pisa.oecd.org>), which rolls everything into a single "capacity":

> "***Mathematics literacy*** *is an individual's capacity*
> *- to identify and understand the role that mathematics plays in the world,*
> *- to make well-founded mathematical judgments and*
> *- to engage in mathematics,*
> *in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen."*

This elusive notion PISA then claims to "measure"!

The PISA "definition" has a certain appeal. But the examples below show that it is far too pretentious to lead to reliable assessment *even in a single classroom*, let alone nationally or internationally. If one tries, the problems used are bound to be culturally biased; the marking schemes are likely to be artificial, and their implementation less than robust! This may help to explain why a country like England, where basic technique is so weak, showed up so well on (the admittedly small scale) PISA 2000, yet performed so poorly in PISA 2006. Sadly, the whole PISA process – problems, methodology, implementation, interpretation, etc. – remains largely "hidden" (in the UK the details are buried in the *Office for National Statistics*, a government agency which is not in the habit of engaging in open academic debate). The neutral observers I know who have tried to make an honest assessment of PISA have all come to the uncomfortable conclusion that there is something seriously amiss at almost all levels of the PISA program.

The examples that follow (which are much simpler than the problems used in PISA) have been chosen to illustrate students' ability to *use* elementary mathematics. These simple tasks reveal basic obstructions to any attempt to "engage effectively with quantitative information in simple settings"; their simplicity makes the observed outcomes more telling; and the weaknesses they reveal are such as to make more complex assessment tasks seem inappropriate. Moreover, they illustrate what has happened in a country which embraced "numeracy" 20 years before those who are now in headlong pursuit of this educational phantasm. Their message is clear:

> those who prefer Salieri to Mozart risk landing up *with nothing of lasting value*.

TIMSS 1995 (Large sample of 13 and 14 year olds - 13 years after Cockcroft!)

P13. *A person's heart is beating 72 times a minute. At this rate, about how many times does it beat in one hour?*

A. 420 000 *B. 42 000* *C.* ***4 200*** *D. 420*

Highest scoring country 86.8%

Average 63.8%

England 44.6%

2003-7 (First year students majoring in mathematics at a good English university)

Q1. Two cyclists *Two cyclists, 42km apart, are heading towards each other. They set out at 8am. At 11am they pass each other. One travels at an average speed of 7.5 km/h. What is the average speed of the other cyclist?*

Successful 67% - 75%

Q2. Tom, Dick and Harry *Tom and Dick take 2 hours to complete a job; Dick and Harry take 3 hours to do the same job; Harry and Tom take 4 hours for the job. How long would all three of them take for the job - working together?*

Successful 0% - 3%

Despite the selective samples (80-180 mathematics students each year), the success rates on the last two problems are remarkably consistent. The third problem is hard, and will no doubt have its critics; but an ability to handle rate problems intelligently should be part of any notion of "quantitative literacy". So the fact that *almost all* mathematics majors think they can model the given problem by solving such equations as "$T + D = 2$" should be highly disturbing.

Mathematics instruction is never as effective as one would like. In countries with strict social expectations and carefully structured teaching, large numbers of students master basic methods rather well (e.g. the highest scoring countries achieved impressive success rates on the TIMSS problems); yet their students often declare a distaste for Mathematics. In countries with a more relaxed social structure, students may say they "like" mathematics, yet perform the most basic procedures in a way which obliges one to ask what such a claim can possibly mean. Nevertheless, the examples I have given are definitely trying to tell us something, and it is time the advocates of *numeracy*, "situated learning", and "Realistic Mathematics Education" tuned in and started to listen. The English infrastructure may be weak, but such examples cannot be simply shrugged off. They illustrate basic failures in students' ability to use mathematics, which do **not** arise because the importance of "numeracy" has been neglected: indeed for 20 years England has experienced little else.

Something has clearly gone very wrong – and perhaps not only in England. But what? Any summary is bound to be inadequate, but the root of the problem in England (and maybe elsewhere) would seem to be that we have lost sight of what constitute the key ideas and methods of elementary mathematics, and have "forgotten" how much time and effort is needed to lay foundations and to develop fluency, precision and flexibility in using these ideas.

In struggling to be a little more precise we enlist an unlikely helper! Antonio Gramsci (1891-1937) was a radical, but enlightened Italian communist between the wars. As a result of his activities, Gramsci spent long periods in prison, and used much of this time for study and writing. His diaries are extensive and fascinating. In his diary for 1932 we read:

> *"The new concept of schools is in its romantic phase, in which the replacement of "mechanical" by "natural" methods has become unhealthily exaggerated. […]*
> *Previously pupils at least acquired a certain baggage of concrete facts.*
> *Now there will no longer be any baggage to put in order […]*
> *The most paradoxical aspect of it all is that this new type of school is advocated as being democratic, while in fact it is destined not merely to perpetuate social differences but to crystallize them in Chinese complexities."*

In "turn of the century" England this "natural-ist" fallacy took many forms, such as:

- a touching faith in "students' own methods", even where this hinders their progress
- a distaste for doing the groundwork of establishing a robust fluency in basic technique
- claims that it is "unnatural not to encourage the use of calculators"
- the idea that one can learn to "estimate" without linking this to **exact** calculation
- the confusion of "algebra" with subjective "pattern spotting"
- the view that "abstraction" is off-putting and can be avoided
- an insistence on embedding problems in "fake contexts", despite the irrelevant "noise" and confusion which then undermines the claimed purpose of the task
- a "behaviourist" belief that mathematics can be reduced to simple "outcomes", which can then be taught and assessed one at a time - forgetting that the hardest, and most rewarding, aspect of learning mathematics is the challenge *to integrate simple steps into effective wholes*.

The overall situation is also complicated by social changes, among which I single out

- the impact of mass education, and
- the fact that within a single generation, the hard-won humano-scientific Enlightenment ideal – that way of knowing and living which struggles to combine respect for truth

with respect for humanity – has been almost swept away by a breathtakingly crass "consumer democracy".

In place of the traditional image of life as an "honest, but rewarding, upward struggle", we have embraced the delusion of an endless, painless downhill "free ride".

In this Humpty Dumpty world, the **consumer** (that is, the student!) is always right – no matter how ignorant s/he may be; everyone is encouraged to have their own opinion, and to use the power of modern communications to express it – a process which converges inexorably to some lowest common denominator. Leadership, and right and wrong, have vanished; in their place we have a potent, but ultimately vacuous, combination of *political opportunism* and *bureaucratic control through accountability*. In such a world there is no such thing as traditional education, and there can be no such thing as "mathematics".

The mental universe of mathematics

Returning to the more restricted world of mathematics education in England, perhaps our biggest mistake has been that we have lost sight of the most basic fact of all - namely that

the world of mathematics is a *mental universe*.

In order to open the minds of ordinary students to the power and flexibility of genuine mathematics

- we must rediscover the fact that mathematics is in some ways inescapably *abstract* from the very beginning, and that effective mathematics teaching has to reflect this fact (in a sensitive way), and
- we must concentrate on identifying – and achieving mastery of – core techniques, which are routinely (and flexibly) linked into multi-step wholes to solve *extended* exercises and problems.

It is also important to apply learned techniques to everyday situations, and to motivate this habit by using suitably chosen "real" problems. But to achieve the kind of mastery and "at-homeness" students need if they are to make mathematical sense of the simplest "real" problems, they must first be able to move around freely inside the mental universe which constitutes elementary mathematics. And if we want students to be 'at-home' in this mental universe, we must ensure they come to experience it, and the associated *mental operations*, as *"real"* in some sense (see (Gardiner, 2007) for the author's attempt to prepare a limited, but significant, group of students to "move around freely inside the universe of elementary mathematics").

Imagination, literacy and the three Rs

The "mental" nature of the "mathematical universe" has one truly liberating consequence (provided one remains faithful to the elusive, but undeniably "objective", character of the discipline) – namely that

elementary mathematics is accessible to anyone with a mind.

And once students' imagination is released, everything becomes possible.

'Tis strange that Things unseen should be Supreme.
The Eye's confined, the Body's pent
In narrow Room; Limbs are of small Extent,
But thoughts are always free.
And as they're best
So can they even in the breast
Rove o'er the World with Liberty;
Can enter Ages, Present be
In any Kingdom, into Bosoms see.
Thoughts, Thoughts can come to Things and view
What Bodies can't approach unto.
They know no Bar, Denial, Limit, Wall:
But have a Liberty to look on all. (Traherne, *Thoughts, I*)

Our concern here is to identify those key ingredients of this "mental universe of mathematics"

(i) which are needed to lay a flexible mathematical foundation for all at ages 5-15;
(ii) which are relevant on the simplest level of *numeracy*; and
(iii) which might represent a modest outline of the kind of features one could include in a more appropriate interpretation of *mathematical literacy* as a desirable adult by-product of school mathematics.

Our tentative ingredients are deliberately simple, and avoid epistemological considerations. However, for even such apparently low level objectives to be achievable, what is taught and learned must respect what Semadeni (Semadeni, 2004) has called "the triple nature of mathematics". That is, the underlying philosophy needs to go beyond the surface features of familiar notation and techniques, to reflect the way mathematics sifts out *deep ideas* (quite different from the "Big Ideas" beloved of many educationists) and transforms them into systems within which we can *think* and *calculate*.

What constitutes numeracy?

As suggested earlier, we do not try to define "numeracy", but simply take it to mean:

a willingness to engage effectively with quantitative information in simple settings.

We list four components which would seem to be necessary in any program to achieve such a "willingness to engage effectively with quantitative information in simple settings". While these include competence in handling certain basic techniques, "numeracy" cannot be reduced to a detailed syllabus. Hence the list included in the third of our four components is deliberately short, and the other three components are in many ways more important.

1. The (not-so-traditional) three Rs: The first component of our attempt to pin down what might be meant by "basic numeracy" is the deliberately provocative *Trinity of Obligations*:

(i) to **remember** (i.e. to learn - if necessary by "rote")
(ii) to **"reckon"** (i.e. to calculate accurately)
(iii) to **reason** (i.e. to think mathematically).

2. Mathematics as the science of exact calculation: The second component - which is *essential* if the other three components are to be effective - is the suggestion that all students should absorb and appreciate the spirit of the statement that, in contrast to the messy "real world",

> **mathematical calculations** (whether numerical, symbolic, logical or geometrical) are *special*, in that they concern "ideal" rather than real objects, and so **are exact**.

3. Techniques: The third component is that *all* students should be expected to achieve mastery of a limited core of basic techniques (some large subset of: multiplication tables, place value and decimals, measures, fractions and ratio, negative numbers, triangles and circles, Pythagoras' theorem, basic trigonometry and similarity, coordinates, linear and quadratic equations, how to handle formulae, straight line graphs and linear functions), which techniques constitute the "background" material in terms of which the other three components can be interpreted.

4. Applications: Whatever list of content and techniques is adopted, these should be used regularly and routinely to handle problems and situations which systematically cultivate the notion that, despite its "ideal" character, one important aspect of elementary mathematics (integer and decimal arithmetic; simple and compound measures; fractions, ratio and simple proportion; the simplest geometrical representations; etc.) is that it is *useful*.

The real test of these four components is whether they can help us to devise strategies which achieve more than we do at present for the majority of students. But, although any such strategy will be judged in terms of the outcomes for its target audience, it is worth stressing that these components are more than merely "utilitarian", and should be seen as part of a profoundly humane education. To remind us that even the first component is included in this spirit, it may be worth sharing three quotations from the writings of George Steiner.

Remembering: *"A cultivation of trained, shared remembrance sets a society in natural touch with its own past [and] safeguards the core of individuality. What is committed to memory and susceptible of recall constitutes the ballast of the self. The pressures of political exaction, the detergent tide of social conformity, cannot tear it from us."*

Reckoning: *"By virtue of mathematics, the stars move out of mythology and into the astronomer's table. And as mathematics settles into the marrow of a science, the concepts of that science, its habits of invention and understanding, become steadily less reducible to those of common language.*
The notion of essential literacy is still rooted in classic values, in a sense of discourse, rhetoric and poetics. But this is ignorance or sloth of imagination. ...
All evidence suggests that the shapes of reality are mathematical, that integral and differential calculus are the alphabet of just perception."

Reasoning: *"How to stick to principle or social aim while facing facts as they are is the peculiar problem for human intelligence in a democratic culture. …*
Anybody can take sides when things are labeled "revolutionary", fascist", "progressive", or "democratic". But what is it we are asked to believe, to consent to, to support? What value is there in opinions that flow from us like the saliva in Pavlov's dogs, at the ringing of a bell?"

What is mathematical literacy?

In seeking to make a useful distinction between the terms "numeracy", "quantitative literacy" and "mathematical literacy", we have proposed at one extreme (following Cockcroft) to interpret "numeracy" as meaning a basic "willingness to engage effectively with quantitative information in simple settings". At the other extreme, echoing the spirit of the familiar notions of "maturity" and "higher literacy", we proposed an interpretation of "mathematical literacy" in more elusive terms as:

a subtle, long-term aspiration involving important insights
into the nature of elementary mathematics and its utility.

(While we shall not discuss "quantitative literacy" here, it might then naturally be interpreted either as a more ambitious "college" version of "numeracy", or as a numerically-oriented version of what we have labelled "mathematical literacy".)

Thus we suggest that the term "mathematical literacy" should be reserved for something distinctively different from "numeracy", namely that it should refer to a deeper *adult residue* of students' experience of school mathematics. Naturally, different adults will emerge from school with such a residue in different measures, but we restrict attention here to those residual insights and competencies which one could reasonably expect – to varying degrees – from a substantial percentage (say 30%-60%) of each cohort.

Some aspects of this "residue" make sense only in the context of *specific learning* (with the depth of this learning varying from one adult to another). Nevertheless, insofar as it makes sense to embrace "mathematical literacy" as a desirable goal for large numbers of adults, it should include the expectation that they:

- know at first hand the most important parts of elementary mathematics
- have achieved complete fluency, automaticity and robust mastery of the basic processes of school mathematics.

However, they should not only have mastered, remembered, and be able to use some core of mathematical techniques; they should also have *reflected* (the fourth "*R*"!) to some extent on *the distinctive character of mathematics*. It is therefore important that they

- have an in-depth experience of at least one "rich" area of elementary mathematics, and
- have extensive experience of grappling with, and solving, simple multi-step problems.

In parallel with this experience of specific content and activity we also need to include those residual "impressions", or general principles, which one would like large numbers of adults to absorb (often unconsciously) from their experience of school mathematics, including:

- a clear distinction between *serious mathematics* and mere tests
- a recognition of, and respect for, the logical character of all mathematics
- an insight into why elementary mathematics is inevitably "abstract" (in some sense)
- a sense of mathematics as "the science of *exact* calculation", and how the basic methods of *exact* calculation can be modified to *approximate* effectively
- an insistence on "meaning", and hence a recognition of the importance of *simplification*
- a recognition that real mathematics begins with multi-step problems
- a recognition of the importance of *connections* between apparently different topics,

and that much of the power of mathematics arises when a simple method from one domain is used in a very different context.

Conclusion

This paper has sought to sound a warning about the hype surrounding the current use of the terms *numeracy*, *quantitative literacy* and *mathematical literacy*, and to attempt an initial analysis which might help us accommodate the important notions of *numeracy* and of *mathematical literacy* within the broader goals of serious mathematics education.

While society has changed dramatically in the course of 5000 or so years of civilization, it would appear that the human mind has evolved rather slowly. The art of introducing young minds to the delights and frustrations of elementary mathematics has also changed far less than we are often led to believe. In the modern era of mass education, a nagging awareness that success in mathematics remains elusive for most pupils therefore compels us to re-consider the assumptions on which current approaches to the teaching of elementary mathematics are based, and to ask whether there might not be some simple alternatives which would allow a larger number of students to taste a little more 'success'.

At the same time, one has to remember that the most likely reason why success remains elusive is that mathematics and mathematics teaching are simply *hard* – in which case the probability that there is some simple, more effective alternative is likely to be rather small. Unfortunately, in any domain where success proves consistently elusive, magic 'solutions' are often proposed, and may even be believed (for a while). In this, mathematics education is no exception: the last 40 years have witnessed a succession of proposed paradigm shifts (understanding before – rather than as a result of – procedures, child-centred discovery learning, technology, "process" not "product", problem-solving, constructivism, outcomes-based education, realistic mathematics education, individualized programs, etc.) each of which, we were assured, would lead to marked improvements; yet in almost all cases, despite interesting exploratory work at the local level for particular groups of pupils and teachers, the eventual outcome has proved disappointing.

The most recent alternatives to traditional school mathematics are reflected in the widespread current use of the terms *numeracy*, *quantitative literacy* and *mathematical literacy*. These expressions now pervade much of the current mathematics education debate – especially in western-style democracies, where politicians, administrators and certain educationists exploit them in their different ways to deconstruct the "elitist" (i.e. hard, and hence politically inconvenient) character of traditional elementary mathematics. Some countries have already changed their national curricula – replacing the traditional label *Mathematics* by *Mathematical Literacy*. Given these pressures, it is important for those in the wider mathematical community – including mathematicians and those who work in mathematics education – to examine critically the claims underlying this global trend to avoid the disappointment of yet another false dawn.

References

Cockcroft. 1982. *Mathematics counts.* HMSO, London (UK).
Crowther. 1959. *15 to 18: Report of the Central Advisory Council for Education (England).* HMSO,

London (UK).

Gardiner, A. 1991. *Discrete mathematics: a cautionary note,* in "Discrete mathematics across the curriculum K-12" (Eds: MJ Kenny and CR Hirsch). NCTM, Reston Va.

Gardiner, A. 1996. *Problem-solving? Or problem solving?* Math. Gazette 80, 143-148.

Gardiner, T. 1988. *Will computers count?* Micromath 4, 31-32.

Gardiner, T. 2007. *Extension mathematics – Book Alpha, Book Beta, Book Gamma (plus Teachers' Book)*. Oxford University Press, Oxford (UK).

HESA. 2008. http://www.hesa.ac.uk, (see "Students and qualifiers" data tables, and compare numbers listed under "Statistics" as "Subject of study" for successive years, bearing in mind the redefinition of "Subjects" from 2002/3).

IFMA. 2008. http://www.ifma.org.il/english/index.html .

Kassell. 2004. http://www/cimt.plymouth.ac.uk/projects/kassell/default.htm .

LMS. 1995. *Tackling the mathematics problem.* London Mathematical Society, Institute of Mathematics and its Applications, Royal Statistical Society; London (UK).

Kline, M. 1973. *Why Johnny can't add: the failure of the new math*. St Martin's Press, New York.

Ma, L. 1999. *Knowing and teaching elementary mathematics*. Lawrence Erlbaum Associates, Mahwah NJ.

MEP. 2008. http://www.cimt.plymouth.ac.uk/projects/mep/default.htm .

PISA. 2008. http://www.pisa.oecd.org

Semadeni, A. 2004. *The triple nature of mathematics: deep ideas, surface representations, formal models.* Proceedings of 10th International Congress of Mathematical Education, Copenhagen.

SingaporeMath.com. 2008. http://www.singaporemath.com/School_Training_s/90.htm .

Smith. 2004. *Making mathematics count.* (<www.mathsinquiry.org.uk>) HMSO, London (UK).

Steen, Lynn A. (ed.) 2001. *Mathematics and democracy, the case for quantitative literacy.* National Council for Education and the Disciplines.

Steen, Lynn A. (ed.) 2004. *Achieving quantitative literacy*. Mathematical Association of America.

TIMSS. 2008. http://timss.bc.edu