Increase Our Learning Horizon with Evolving Technology

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Abstract.
In this short note, we demonstrate how evolving technological tools can allow us to enhance our learning horizon (wherever learners' levels are) in mathematics. We will use examples to show Dynamic Geometry (DG) and its animation and numerical approximations can make mathematics accessible to more learners. In the meantime, problems can be made challenging when learners attempt to generalize the result by using a Computer Algebra System (CAS). Finally, we show some results valid in two dimensions can be extended to three dimensional ones too.

1. Mathematics contents can be made accessible and challenging

The parametric graph of a curve is an important concept when learners expand their knowledge of the graph of a function. The example below is to ask when a circle is drawn, what could we get out of the parametric equations.

Example 1. A circle is drawn (arbitrarily) using a Dynamic Geometry System, which we use ClassPad (see [1]) for demonstration. Then pick a point P=(x(t),y(t)) on the circle, where t is ranging from 0 to 2π . What are the graphs of (t, x(t)) and (t, y(t)) respectively.

We start with the following circle, which has center C= [-1.35,-0.225] and radius r=1.179248.

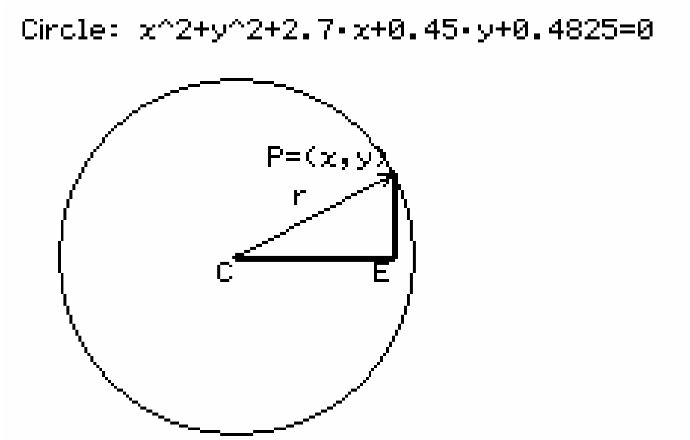
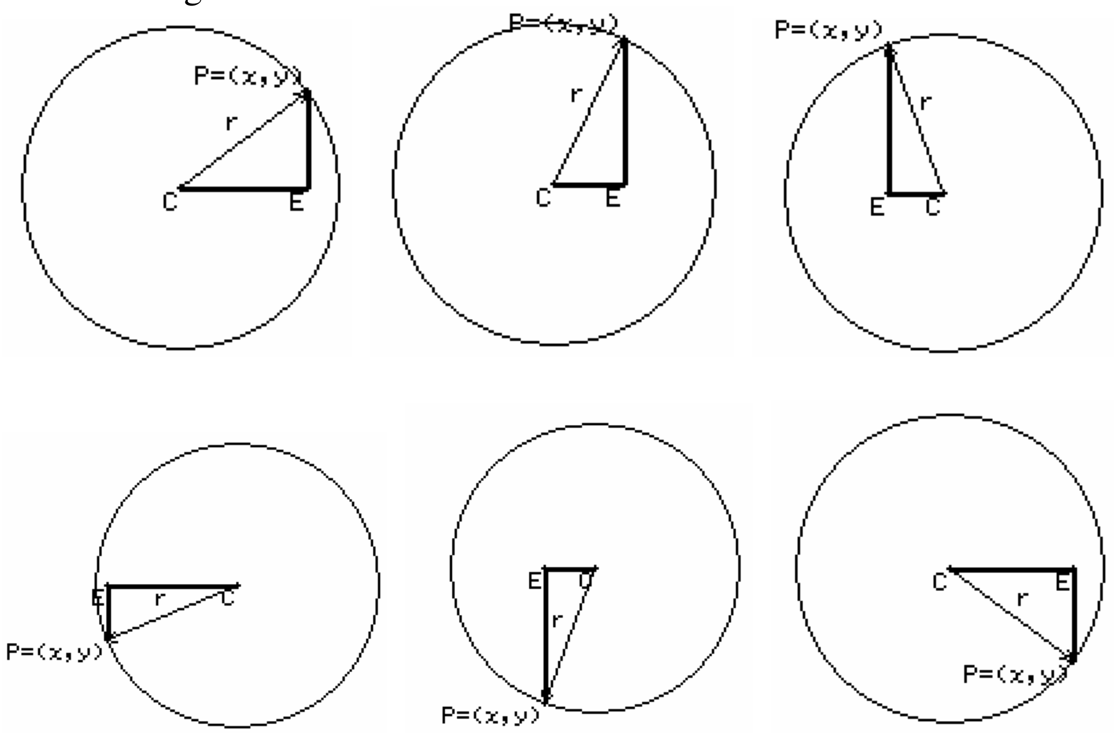


Figure 1.

We pick a point P=(x,y) and let it animate once on the circle and obtained the following screen shots:



Figures 2-7.

In the mean time, we collect the (t,x(t)) and (t,y(t)) respectively and we show the partial list as follows:

Direction	x	y
0	-0.17075	-0.225
0.161107	-0.18602	-0.03584
0.322215	-0.23144	0.148430
0.483322	-0.30583	0.323024
0.644429	-0.40726	0.483424
0.805537	-0.53311	0.625476
0.966644	-0.68011	0.745502
1.127751	-0.84447	0.840392
1.288859	-1.02191	0.907689
1.449966	-1.20786	0.945650
1.611073	-1.39748	0.953291
1.772180	-1.58588	0.930416
1.933288	-1.76817	0.877616
2.094395	-1.93962	0.796258
2.255502	-2.09581	0.688451
2.416610	-2.23268	0.556986
2.577717	-2.34669	0.405268
2.738824	-2.43488	0.237226
2.899932	-2.49498	0.057212
3.061039	-2.52542	-0.13011
3.222146	-2.52542	-0.31989
3.383254	-2.49498	-0.50721
3.544361	-2.43488	-0.68723

Figure 8.

Finally, we drag the columns ‘Direction’ (which is the angle t) and x back to the Figure 1, we see the following curve in addition to the circle. Can you name that curve?

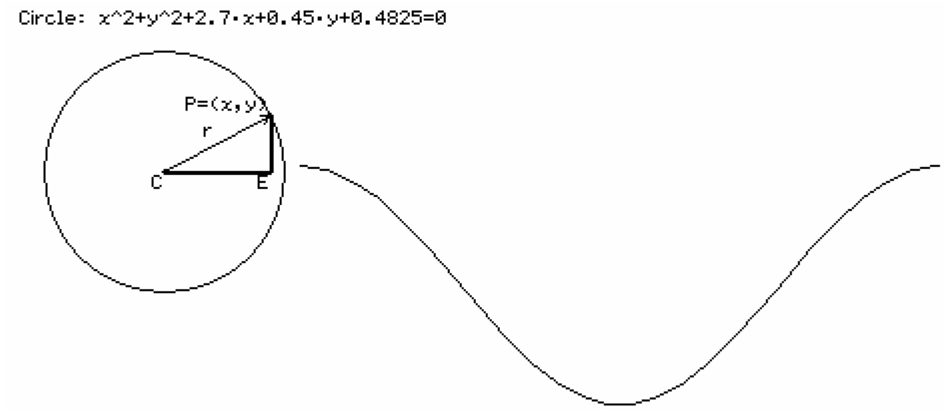


Figure 9.

Similarly, we drag the columns ‘Direction’ (which is the angle t) and y back to the Figure 1, we see the following curve in addition to the circle. Can you name that curve?

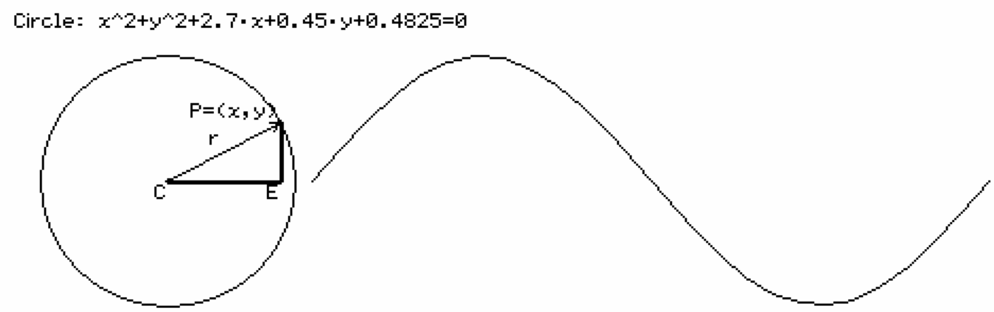


Figure 10.

Readers can link to a video clip which demonstrates the processes above (see [2]). The above exercise is to motivate students to explore the parametric equation of the form [x(t),y(t)]=[a+r*cos(t),b+r*sin(t)], where (a,b) is the center of the circle of radius r. Experienced learners know the above parametric equation represent a circle. However, when we pose the question reversely, the problem becomes more interesting.

2. Integrating Dynamic Geometry with CAS

W may use the numerical capability of DG to approximate a solution. This makes mathematics accessible to more students before one gets into introducing the theory behind a concept. So this type of probem can be introduced to learners with minimum knowledge. In the mean time, the problem can be re-introduced again and becomes challenging when we use a CAS to verify if our conjecture is true. This spiral way of learning process makes students understand the connection between the old knowledge they learned from the past and make a good argument why new knowlege needs to be introduced.

Example 2. We are given a line L of the form $y = mx + b$ and a curve S (see Figure 11 below), find the reflection of the curve S respective to the line L.

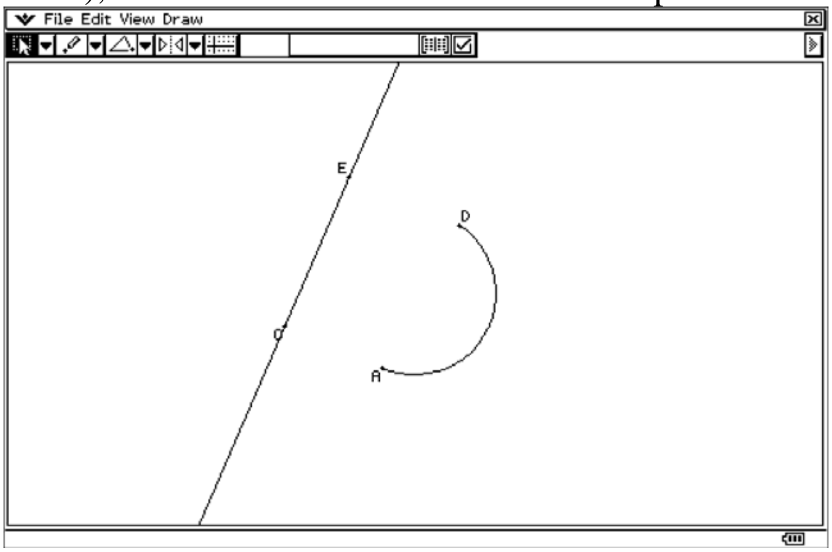


Figure 11.

Assume the general case for finding the inverse of $[x(t), y(t)]$ with respect to $y = mx + b$. We first set $\theta = \tan^{-1} m$. We call the reflection of $[x(t), y(t)]$ with respect to $y = mx + b$ to be $[p(t), q(t)]$.

It can be proved that

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}.$$

Example 3. Find the reflection of $\begin{bmatrix} 2\cos t - \cos(2t) \\ 2\sin t - \sin(2t) \end{bmatrix}$ with respect to $y = 2x + 1$.

We set $\theta = \tan^{-1} 2$ and we obtain $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ to be

$$\begin{bmatrix} (2\cdot\cos(t)-\cos(2\cdot t))\cdot\cos\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)+(2\cdot\sin(t)-\sin(2\cdot t)-1)\cdot\sin\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right) \\ (2\cdot\cos(t)-\cos(2\cdot t))\cdot\sin\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)-(2\cdot\sin(t)-\sin(2\cdot t)-1)\cdot\cos\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)+1 \end{bmatrix}$$

.,

We plot $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ (thick) together with $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ (thin) below:

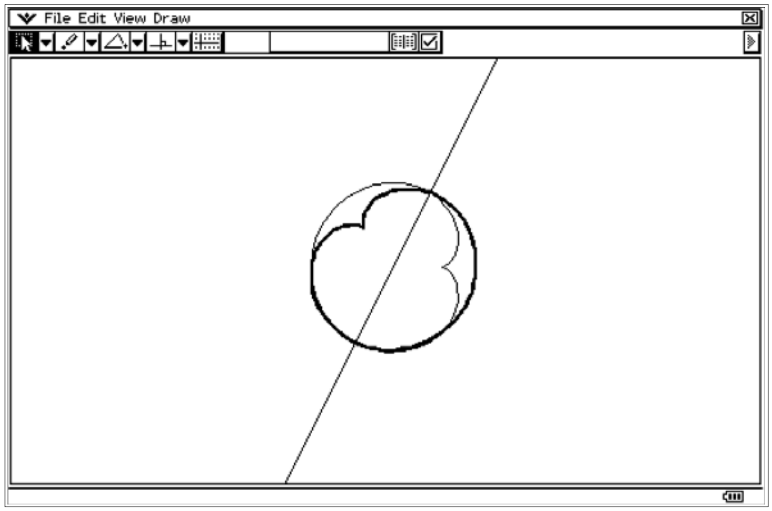


Figure 12.

Example 4. (Hypocyloid) Find the reflection of $\begin{bmatrix} (a-b)\cos t + b\cos((\frac{a}{b}-1)t) \\ (a-b)\sin t + b\sin((\frac{a}{b}-1)t) \end{bmatrix}$ $(a=3$ and $b=1)$ with respect to $y=2x+1.$

We set $\theta = \tan^{-1} 2$ and we obtain $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ to be

$$\begin{bmatrix} (2\cdot\cos(t)+\cos(2\cdot t))\cdot\cos\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)+(2\cdot\sin(t)-\sin(2\cdot t)-1)\cdot\sin\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right) \\ (2\cdot\cos(t)+\cos(2\cdot t))\cdot\sin\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)-(2\cdot\sin(t)-\sin(2\cdot t)-1)\cdot\cos\left(2\cdot\left(-\tan^{-1}\left(\frac{1}{2}\right)+\frac{\pi}{2}\right)\right)+1 \end{bmatrix}$$

We plot $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ (thick) together with $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ (thin) below:

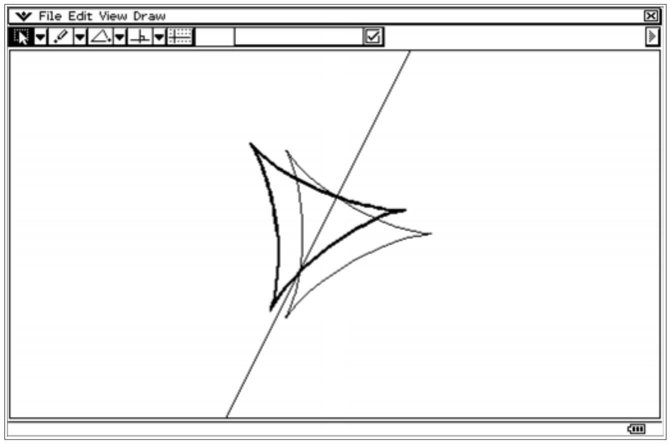


Figure 13.

3. Going from 2D to 3D

The example below shows how we can link the idea of Lagrange Multipliers, its geometric interpretation with linear indecency of Linear Algebra together with the help of Dynamic Geometry Software packages.

Example 5. We are given three curves in the plane, see $C1, C2$ and $C3$ below. We need to find points A, B, and C on $C1, C2$ and $C3$ respectively so that the distance $AB + AC$ achieves its minimum.

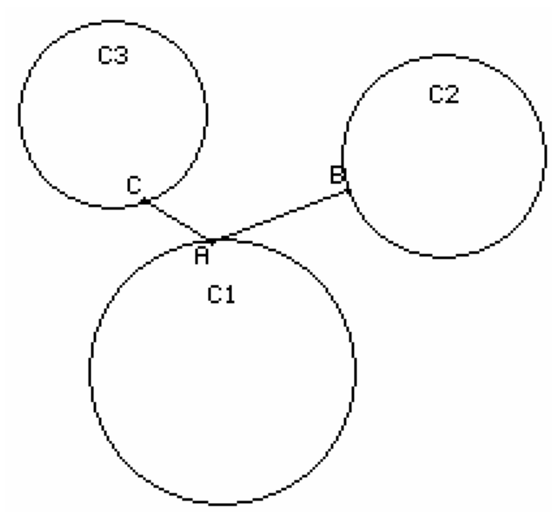


Figure 14.

We note that we may use a DG software to make conjectures of when the minimum distance should occur, but with some knowledge of Linear Algebra and Multi-variable Calculus, it is not hard to make the following observations:

- AB should be parallel to the normal vector of the curve C2 at B.
- AC should be parallel to the normal vector of the curve C3 at C.
- We should place the points A, B and C (on C1, C2 and C3 respectively) so that **the normal vector of C1 at A=linear combination of AB and AC.**
- The above observation is precisely what we expect from applying technique of Lagrange Multipliers.

We extend the concept to 3D where we can use a 3D DG such as Cabri 3D to explore.

Example 6. We are given four surfaces in the space, represented by the orange surface, called $S1$; yellow surface, called $S2$; blue surface called $S3$ and the purple surface, called $S4$ respectively. We want to find points A, B, C and D on $S1, S2, S3$ and $S4$ respectively so that the distance $AB + AC + AD$ achieves its minimum.

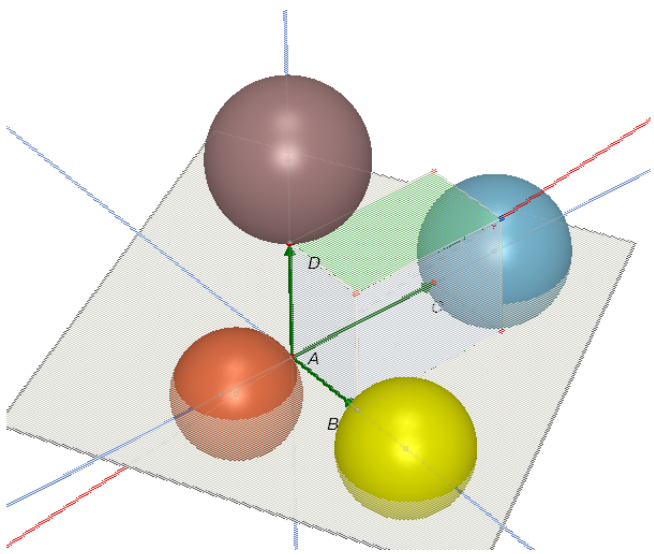


Figure 15.

It is natural to have the following observations:

- AB should be parallel to the normal vector of the surface S2 at B.
- AC should be parallel to the normal vector of the surface S3 at C.
- AD should be parallel to the normal vector of the surface S4 at D.
- We should place the points A, B, C and D (on S1, S2, S3 and S4 respectively) so that

The normal vector of S1 at A=linear combination of AB, AC and AD.

Certainly the above concept can be extended to any finite dimensions too.

4. Conclusion

Technology definitely can not solves all our problems. Implementing technological tools into teaching and learning is a not trivial task and it will be an on-going discussion issue many years to come. One of the issues that we heard often is that many students have lost confidence or interests before entering universities because of the deficency in algebraic manipulation skills. It may be possible to build a curriculum around mathematical programs when teachers introduces DG so mathematics is more accessible to students at younger age and inspire students why Algebra is needed. If mathematics problems are chosen properly, same problems they encountered in the past can be re-solved again using their added knowledge when students entering universities. From the examples we see above, DG indeed can make mathematics more accessible, and when a CAS is used to prove results analytically, mathematics is challenging too.

Acknowledgement.
Author would like to thank Jean-Jacques Dahan of France of creating the Cabri 3D (see [3]) file for Figure 15.

References

[1] ClassPad, a product of CASIO Computer LTd, http://classpad.net or http://classpad.org.
[2] A video clip, http://mathandtech.org/CASIO_Video/Trig_Shifting2/Trig_Shifting2.html.
[3] Cabri 3D, a product of Cabrilog, http://cabri.com.